Execution-time opacity problems in one-clock parametric timed automata

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Abstract

Parametric timed automata (PTAs) extend the concept of timed automata, by allowing timing delays not only specified by concrete values but also by parameters, allowing the analysis of systems with uncertainty regarding timing behaviors. The full execution-time opacity is defined as the problem in which an attacker must never be able to deduce whether some private location was visited, by only observing the execution time. The problem of full ET-opacity emptiness (i.e., the emptiness over the parameter valuations for which full execution-time opacity is satisfied) is known to be undecidable for general PTAs. We therefore focus here on one-clock PTAs with integer-valued parameters over dense time. We show that the full ET-opacity emptiness is undecidable for a sufficiently large number of parameters, but is decidable for a single parameter, and exact synthesis can be effectively achieved. Our proofs rely on a novel construction as well as on variants of Presburger arithmetics. We finally prove an additional decidability result on an existential variant of execution-time opacity.

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1 Introduction

As surveyed in [10], for some systems, private information may be deduced simply by observation of public information. For example, it may be possible to infer the content of some memory space from the access times of a cryptographic module.

The notion of *opacity* [23, 12] concerns information leaks from a system to an attacker; that is, it expresses the power of the attacker to deduce some secret information based on some publicly observable behaviors. If an attacker observing a subset of the actions cannot deduce whether a given sequence of actions has been performed, then the system is opaque. Time particularly influences the deductive capabilities of the attacker. It has been shown in [16] that it is possible for models that are opaque when timing constraints are omitted, to be non-opaque when those constraints are added to the models.

For this reason, the notion is extended to *timed* opacity in [14], where the attacker can also observe time. The input model is timed automata (TAs) [1], a formalism extending finite-state automata with real-time variables called *clocks*. It is proved in [14] that this version of timed opacity is undecidable for TAs.

In [7], a less powerful version of opacity is proposed, where the attacker has access only to the system execution time and aims at deducing whether a private location was visited during the system execution. This version of timed opacity is called *execution-time opacity (ET-opacity)*. Two main problems are considered in [7]: 1) the existence of at least one execution time for which the system is ET-opaque ($\exists$-*ET-opacity*), and 2) whether *all* execution times are such that the system is ET-opaque (called *full ET-opacity*). These two notions of opacity are proved to be decidable for TAs [5]. In the same works, the authors then extend ET-opacity to parametric timed automata (PTAs) [2]. PTAs are an extension of TAs where timed constraints can be expressed with timing parameters instead of integer constants, allowing to model uncertainty or lack of knowledge. The two problems come with two flavors: 1) *emptiness* problems: whether the set of parameter valuations guaranteeing a given version of opacity is empty or not, and 2) *synthesis* problems: synthesize all parameter valuations for which a given version of opacity holds. Both emptiness problems ∃OE (∃-ET-opacity emptiness) and FOE (full-ET-opacity emptiness) are proved undecidable for PTAs, while decidable subclasses are exhibited [7, 5]. A semi-algorithm (i.e., that may not terminate, but is correct if it does) is provided to solve full ET-opacity synthesis (hereafter FOS) in [7].

1.1 Contributions

We address here full-ET-opacity emptiness (FOE) and synthesis (FOS), and ∃-ET-opacity emptiness (∃OE) and synthesis (∃OS), for PTAs with integer-valued parameters over dense time with the following theoretical main contributions:

1. We prove that **FOE is undecidable** (Corollary 29) for PTAs with a single clock and a sufficiently large number of parameters.
2. We prove in contrast that **FOE is decidable** (Corollary 30) for PTAs with a single clock and a single parameter.
3. We prove that **∃OE is decidable** (Theorem 31) for PTAs with a single clock and arbitrarily many parameters. We also exhibit a better complexity for a single parameter over discrete time (Theorem 33).

Our contributions are summarized in Table 1. In order to prove these results, we improve on the semi-algorithm from [7] for ∃OS and provide one for FOS. These solutions are based on the novel notion of *parametric execution times* (PET). The PET of a PTA is the total elapsed time and associated parameter valuations on all paths between two given locations. We provide a semi-algorithm for the computation of PET, and then show how to resolve ∃OS and FOS problems by performing set operations on PET of two complementary subsets of the PTA where we respectively consider only private paths and only non-private paths.

We then solve the full ET-opacity emptiness (FOE) problem for PTAs with 1 clock and 1 parameter, by rewriting the problems in a parametric variant of Presburger arithmetic. This is done by 1) providing a sound and complete method for encoding infinite PET for PTAs with 1 clock and arbitrarily many parameters over dense time; and 2) translating them into parametric semi-linear sets that can be handled using [22]. With these ingredients, we notably prove that: 1) FOE is undecidable in general for PTAs with 1 clock and sufficiently many parameters. This is done by reducing a known undecidable problem of parametric Presburger arithmetic (which undecidability comes from Hilbert 10th problem) to the FOE problem in this context. 2) ∃OE is decidable for PTAs with 1 clock and arbitrarily many parameters. This is done by reducing ∃OE to the existential fragment of Presburger arithmetic with divisibility, known to be decidable.

1.2 Related works

The negative result of [14] leaves hope for decidability only by modifying the problem (as in [7, 5]), or by restraining the model. In [26, 27], (initial state) opacity is shown to be decidable on a restricted subclass of TAs called real-time automata [15]. In [3], a notion of *timed bounded opacity*, where the secret has an expiration date, and over a time-bounded framework, is proved decidable.

In [7], ∃-ET-opacity synthesis (∃OS) is solved using a semi-algorithm. The method is based on a self-composition of the PTA with m parameters and n clocks, where the resulting model is composed of $m + 1$ parameters and $2n + 1$ clocks. The method terminates if the symbolic state space of this self-composition is finite. Our work proposes in contrast an approach based on set operations on parametric execution times (PET) of both complementary subsets of the PTA where we respectively consider only private paths and only non-private paths. Those submodels are each composed of $m+1$ parameters and $n+1$ clocks. Our new method terminates if the symbolic state spaces of both submodels are finite. Another improvement is that the method described here also supports full timed opacity synthesis (FOS).

The reachability emptiness problem (i.e., the emptiness over the valuations set for which a given target location is reachable) is known to be undecidable in general since [2]. The rare decidable settings require a look at the number of parametric clocks (i.e., compared at least once in a guard or invariant to a parameter), non-parametric clocks and parameters; throughout this paper, we denote these 3 numbers using a triple (pc, npc, p). Reachability emptiness is decidable for $(1,*,*)$-PTAs ("$*$" denotes "arbitrarily many" for decidable cases, and "sufficiently many" for undecidable cases) over discrete time [2] or dense time with integer-valued parameters [9], for $(1,0,*)$-PTAs over dense time over rational-valued parameters [8], and for $(2,*,1)$-PTAs over discrete time [13, 17]; and it is undecidable for $(3,*,1)$-PTAs over discrete or dense time [9], and for $(1,3,1)$-PTAs over dense time only for rational-valued parameters [24]. See [4] for a complete survey as of 2019.

Section 2 recalls the necessary preliminaries. Section 3 introduces one of our main technical proof ingredients, i.e., the definition of PET, and PET-based semi-algorithms for ∃OS and FOS. Section 4 considers the FOE problem over $(1,0,*)$-PTAs (undecidable) and $(1,0,1)$-PTAs (decidable). Section 5 proves decidability of ∃OE for $(1,0,*)$-PTAs. We also give a better complexity for $(1,0,1)$-PTAs over discrete time. Section 6 concludes.

2 Preliminaries

We let $\mathbb{T}$ be the domain of the time, which will be either non-negative reals $\mathbb{R}_{\geq 0}$ (continuous-time semantics) or naturals $\mathbb{N}$ (discrete-time semantics). Unless otherwise specified, we assume $\mathbb{T} = \mathbb{R}_{\geq 0}$.

Clocks are real-valued variables that all evolve over time at the same rate. We assume a set $\mathbb{X} = \{x_1, \dots, x_H\}$ of *clocks*. A *clock valuation* is a function $\mu : \mathbb{X} \to \mathbb{T}$. We write $\vec{0}$ for the clock valuation assigning 0 to all clocks. Given a constant $\gamma \in \mathbb{T}$, $\mu + \gamma$ denotes the valuation s.t. $(\mu+\gamma)(x) = \mu(x) + \gamma$, for all $x \in \mathbb{X}$. Given $R \subseteq \mathbb{X}$, we define the *reset* of a valuation μ, denoted by $[\mu]_R$, as follows: $[\mu]_R(x) = 0$ if $x \in R$, and $[\mu]_R(x) = \mu(x)$ otherwise.

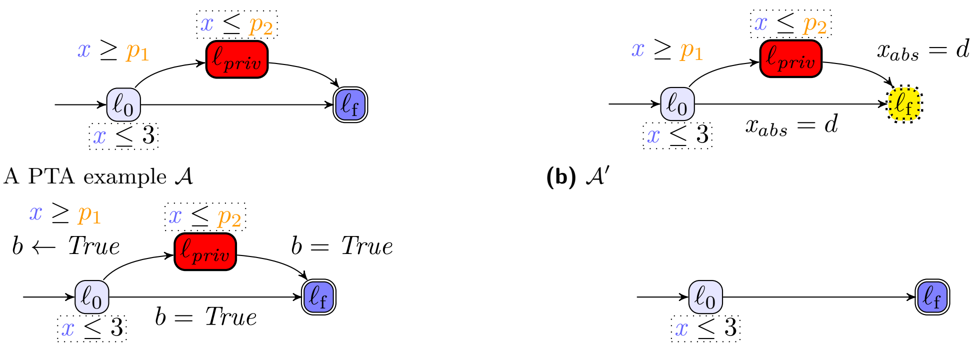
A *(timing) parameter* is an unknown integer-valued constant of a model. We assume a set $\mathbb{P} = \{p_1, \dots, p_M\}$ of *parameters*. A *parameter valuation* v is a function $v : \mathbb{P} \to \mathbb{N}$.

We assume $\bowtie \in \{<, \leq, =, \geq, >\}$. A *clock guard* C is a conjunction of inequalities over $\mathbb{X} \cup \mathbb{P}$ of the form $x \bowtie \sum_{1 \leq i \leq M} \alpha_i p_i + \gamma$, with $p_i \in \mathbb{P}$, and $\alpha_i, \gamma \in \mathbb{Z}$. Given C, we write $\mu \models v(C)$ if the expression obtained by replacing each x with $\mu(x)$ and each p with $v(p)$ in C evaluates to true.

2.1 Parametric timed automata

Parametric timed automata (PTAs) extend TAs with parameters within guards and invariants in place of integer constants [2]. We also add to the standard definition of PTAs a special private location, which will be used to define our subsequent opacity concepts.

▶ **Definition 1** (PTA [2]). *A parametric timed automaton (PTA) [2] $\mathcal{A}$ is a tuple $\mathcal{A} = (\Sigma, L, \ell_0, \ell_{priv}, \ell_f, \mathbb{X}, \mathbb{P}, I, E)$, where: 1) Σ is a finite set of actions; 2) L is a finite set of locations; 3) $\ell_0 \in L$ is the initial location; 4) $\ell_{priv} \in L$ is a special private location; 5) $\ell_f \in L$ is the final location; 6) $\mathbb{X}$ is a finite set of clocks; 7) $\mathbb{P}$ is a finite set of parameters;*



(a) A PTA example $\mathcal{A}$
(b) $\mathcal{A}'$
(c) $\mathcal{A}_{\ell_f}^{\ell_{priv}}$
(d) $\mathcal{A}_{\ell_f}^{\neg\ell_{priv}}$

Figure 1 A PTA example and its transformed versions. The yellow dotted location is urgent.

8) I is the invariant, assigning to every $\ell \in L$ a clock guard $I(\ell)$ (called invariant*); 9) E is a finite set of edges $e = (\ell, g, a, R, \ell')$ where $\ell, \ell' \in L$ are the source and target locations, $a \in \Sigma$, $R \subseteq \mathbb{X}$ is a set of clocks to be reset, and g is a clock guard.*

Given a parameter valuation v, we denote by $v(\mathcal{A})$ the non-parametric structure where all occurrences of a parameter p_i have been replaced by $v(p_i)$.

▶ **Definition 2** (Reset-free PTA)**.** *A* reset-free PTA *$\mathcal{A} = (\Sigma, L, \ell_0, \ell_{priv}, \ell_f, \mathbb{X}, \mathbb{P}, I, E)$ is a PTA where $\forall\, (\ell, g, a, R, \ell') \in E$, $R = \emptyset$.*

▶ **Example 3.** Consider the PTA $\mathcal{A}$ in Figure 1a. It has three locations, one clock and two parameters (actions are omitted). "$x \leq p_2$" is the invariant of ℓ_{priv}, and the transition from ℓ_0 to ℓ_{priv} has guard "$x \geq p_1$". In this example, x is never reset, and therefore $\mathcal{A}$ happens to be reset-free.

▶ **Definition 4** (Semantics of a timed automaton (TA) [1])**.** *Given a PTA $\mathcal{A} = (\Sigma, L, \ell_0, \ell_{priv}, \ell_f, \mathbb{X}, \mathbb{P}, I, E)$ and a parameter valuation v, the semantics of the TA $v(\mathcal{A})$ is given by the timed transition system (TTS) [18] $\mathfrak{T}_{v(\mathcal{A})} = (\mathfrak{S}, \mathfrak{s}_0, \Sigma \cup \mathbb{R}_{\geq 0}, \rightarrow)$, with*

1. $\mathfrak{S} = \{(\ell, \mu) \in L \times \mathbb{R}_{\geq 0}^H \mid \mu \models I(\ell)v\}$, $\mathfrak{s}_0 = (\ell_0, \vec{0})$,
2. *$\rightarrow$ consists of the discrete and (continuous) delay transition relations:*
   a. *discrete transitions: $(\ell, \mu) \overset{e}{\mapsto} (\ell', \mu')$, if $(\ell, \mu), (\ell', \mu') \in \mathfrak{S}$, and there exists $e = (\ell, g, a, R, \ell') \in E$, such that $\mu' = [\mu]_R$, and $\mu \models v(g)$.*
   b. *delay transitions: $(\ell, \mu) \overset{\gamma}{\mapsto} (\ell, \mu + \gamma)$, with $\gamma \in \mathbb{R}_{\geq 0}$, if $\forall \gamma' \in [0, \gamma], (\ell, \mu + \gamma') \in \mathfrak{S}$.*

Moreover we write $(\ell, \mu) \xrightarrow{(\gamma, e)} (\ell', \mu')$ for a combination of a delay and discrete transition if $\exists \mu'' : (\ell, \mu) \overset{\gamma}{\mapsto} (\ell, \mu'') \overset{e}{\mapsto} (\ell', \mu')$.

Given a TA $v(\mathcal{A})$ with concrete semantics $(\mathfrak{S}, \mathfrak{s}_0, \Sigma \cup \mathbb{R}_{\geq 0}, \rightarrow)$, we refer to the states of $\mathfrak{S}$ as the *concrete states* of $v(\mathcal{A})$. A *run* of $v(\mathcal{A})$ is an alternating sequence of concrete states of $v(\mathcal{A})$ and pairs of edges and delays starting from the initial state $\mathfrak{s}_0$ of the form $(\ell_0, \mu_0), (d_0, e_0), (\ell_1, \mu_1), \cdots$ with $i = 0, 1, \ldots$, $e_i \in E$, $d_i \in \mathbb{R}_{\geq 0}$ and $(\ell_i, \mu_i) \xrightarrow{(d_i, e_i)} (\ell_{i+1}, \mu_{i+1})$.

Given a state $\mathfrak{s} = (\ell, \mu)$, we say that $\mathfrak{s}$ is *reachable* in $v(\mathcal{A})$ if $\mathfrak{s}$ appears in a run of $v(\mathcal{A})$. By extension, we say that ℓ is reachable in $v(\mathcal{A})$; and by extension again, given a set L_{target} of locations, we say that L_{target} is reachable in $v(\mathcal{A})$ if there exists $\ell \in L_{target}$ such that ℓ is reachable in $v(\mathcal{A})$.

Given a finite run $\rho : (\ell_0, \mu_0), (d_0, e_0), (\ell_1, \mu_1), \cdots, (d_{i-1}, e_{i-1}), (\ell_\mathrm{n}, \mu_n)$, the *duration* of ρ is $dur(\rho) = \sum_{0 \leq i \leq n-1} d_i$. We also say that ℓ_n is reachable in time $dur(\rho)$.

Let us now recall the symbolic semantics of PTAs (see e.g., [19]). We first define operations on constraints. A *linear term* over $\mathbb{X} \cup \mathbb{P}$ is of the form $\sum_{1 \leq i \leq H} \alpha_i x_i + \sum_{1 \leq j \leq M} \beta_j p_j + \gamma$, with $x_i \in \mathbb{X}$, $p_j \in \mathbb{P}$, and $\alpha_i, \beta_j, \gamma \in \mathbb{Z}$. A *constraint* $\mathbf{C}$ (i.e., a convex polyhedron) over $\mathbb{X} \cup \mathbb{P}$ is a conjunction of inequalities of the form $lt \bowtie 0$, where lt is a linear term. Given a parameter valuation v, $v(\mathbf{C})$ denotes the constraint over $\mathbb{X}$ obtained by replacing each parameter p in $\mathbf{C}$ with $v(p)$. Likewise, given a clock valuation μ, $\mu(v(\mathbf{C}))$ denotes the expression obtained by replacing each clock x in $v(\mathbf{C})$ with $\mu(x)$. We write $\mu \models v(\mathbf{C})$ whenever $\mu(v(\mathbf{C}))$ evaluates to true. We say that v *satisfies* $\mathbf{C}$, denoted by $v \models \mathbf{C}$, if the set of clock valuations satisfying $v(\mathbf{C})$ is nonempty. We say that $\mathbf{C}$ is *satisfiable* if $\exists \mu, v$ s.t. $\mu \models v(\mathbf{C})$. We define the *time elapsing* of $\mathbf{C}$, denoted by $\mathbf{C}^\nearrow$, as the constraint over $\mathbb{X}$ and $\mathbb{P}$ obtained from $\mathbf{C}$ by delaying all clocks by an arbitrary amount of time. That is, $\mu' \models v(\mathbf{C}^\nearrow)$ if $\exists \mu : \mathbb{X} \to \mathbb{R}_{\geq 0}, \exists \gamma \in \mathbb{R}_{\geq 0}$ s.t. $\mu \models v(\mathbf{C}) \land \mu' = \mu + \gamma$. Given $R \subseteq \mathbb{X}$, we define the *reset* of $\mathbf{C}$, denoted by $[\mathbf{C}]_R$, as the constraint obtained from $\mathbf{C}$ by resetting the clocks in R to 0, and keeping the other clocks unchanged. That is,

$$\mu' \models v([\mathbf{C}]_R) \text{ if } \exists \mu : \mathbb{X} \to \mathbb{R}_{\geq 0} \text{ s.t. } \mu \models v(\mathbf{C}) \land \forall x \in \mathbb{X} \begin{cases} \mu'(x) = 0 & \text{if } x \in R \\ \mu'(x) = \mu(x) & \text{otherwise.} \end{cases}$$

We denote by $\mathbf{C}{\downarrow_{\mathbb{P}}}$ the projection of $\mathbf{C}$ onto $\mathbb{P}$, i.e., obtained by eliminating the variables not in $\mathbb{P}$ (e.g., using Fourier-Motzkin [25]).

▶ **Definition 5** (Symbolic state). *A symbolic state is a pair $(\ell, \mathbf{C})$ where $\ell \in L$ is a location, and $\mathbf{C}$ its associated parametric zone.*

▶ **Definition 6** (Symbolic semantics). *Given a PTA $\mathcal{A} = (\Sigma, L, \ell_0, \ell_{priv}, \ell_\mathrm{f}, \mathbb{X}, \mathbb{P}, I, E)$, the symbolic semantics of $\mathcal{A}$ is the labeled transition system called* parametric zone graph $\mathbf{PZG}(\mathcal{A}) = (E, \mathbf{S}, \mathbf{s}_0, \Rightarrow)$, *with*

- $\mathbf{S} = \{(\ell, \mathbf{C}) \mid \mathbf{C} \subseteq I(\ell)\}$, $\mathbf{s}_0 = \big(\ell_0, (\bigwedge_{1 \leq i \leq H} x_i = 0)^\nearrow \land I(\ell_0)\big)$, *and*
- $\big((\ell, \mathbf{C}), e, (\ell', \mathbf{C}')\big) \in \Rightarrow$ *if* $e = (\ell, g, a, R, \ell') \in E$ *and*

$$\mathbf{C}' = \big([(\mathbf{C} \land g)]_R \land I(\ell')\big)^\nearrow \land I(\ell') \textit{ with } \mathbf{C}' \textit{ satisfiable.}$$

That is, in the parametric zone graph, nodes are symbolic states, and arcs are labeled by *edges* of the original PTA.

2.2 Reachability synthesis

We use reachability synthesis to solve the problems defined in Section 2.3. This procedure, called EFsynth, takes as input a PTA $\mathcal{A}$ and a set of target locations L_{target}, and attempts to synthesize all parameter valuations v for which L_{target} is reachable in $v(\mathcal{A})$. EFsynth$(\mathcal{A}, L_{target})$ was formalized in e.g., [20] and is a procedure that may not terminate, but that computes an exact result (sound and complete) if it terminates.

2.3 Execution-time opacity problems [5]

Given a TA $v(\mathcal{A})$ and a run ρ, we say that ℓ_{priv} is *visited on the way to ℓ_f in ρ* if ρ is of the form $\ell_0, \mu_0), (d_0, e_0), (\ell_1, \mu_1), \cdots, (\ell_\mathrm{m}, \mu_m), (d_m, e_m), \cdots (\ell_\mathrm{n}, \mu_n)$ for some $m, n \in \mathbb{N}$ such that $\ell_\mathrm{m} = \ell_{priv}$, $\ell_\mathrm{n} = \ell_\mathrm{f}$ and $\forall 0 \leq i \leq n-1, \ell_\mathrm{i} \neq \ell_\mathrm{f}$. We denote by $Visit^{priv}(v(\mathcal{A}))$ the set of those runs, and refer to them as *private* runs. We denote by $DVisit^{priv}(v(\mathcal{A}))$ the set of all the durations of these runs.

Conversely, we say that ℓ_{priv} is *avoided on the way to* ℓ_f *in* ρ if ρ is of the form $(\ell_0, \mu_0), (d_0, e_0), (\ell_1, \mu_1), \cdots, (\ell_n, \mu_n)$ with $\ell_n = \ell_f$ and $\forall 0 \leq i < n, \ell_i \notin \{\ell_{priv}, \ell_f\}$. We denote the set of those runs by $Visit^{\overline{priv}}(v(\mathcal{A}))$, referring to them as *public* runs, and by $DVisit^{\overline{priv}}(v(\mathcal{A}))$ the set of all the durations of these public runs. Therefore, $DVisit^{priv}(v(\mathcal{A}))$ (resp. $DVisit^{\overline{priv}}(v(\mathcal{A}))$) is the set of all the durations of the runs for which ℓ_{priv} is visited (resp. avoided) on the way to ℓ_f. These concepts can be seen as the set of execution times from the initial location ℓ_0 to the final location ℓ_f while visiting (resp. not visiting) a private location ℓ_{priv}. Observe that, from the definition of the duration of a run, this "execution time" does not include the time spent in ℓ_f.

We now recall formally the concept of "execution-time opacity (ET-opacity) for a set of durations (or execution times) D": a system is *ET-opaque for execution times D* whenever, for any duration in D, it is not possible to deduce whether the system visited ℓ_{priv} or not.

▶ **Definition 7** (Execution-time opacity (ET-opacity) for D)**.** *Given a TA $v(\mathcal{A})$ and a set of execution times D, we say that $v(\mathcal{A})$ is* execution-time opaque (ET-opaque) for execution times D *if* $D \subseteq (DVisit^{priv}(v(\mathcal{A})) \cap DVisit^{\overline{priv}}(v(\mathcal{A})))$.

In the following, we will be interested in the *existence* of such an execution time. We say that a TA is $\exists$-ET-opaque if it is ET-opaque for a non-empty set of execution times.

▶ **Definition 8** ($\exists$-ET-opacity)**.** *A TA $v(\mathcal{A})$ is* $\exists$-ET-opaque *if* $(DVisit^{priv}(v(\mathcal{A})) \cap DVisit^{\overline{priv}}(v(\mathcal{A}))) \neq \emptyset$.

In addition, a system is *fully ET-opaque* if, for any possible measured execution time, an attacker is not able to deduce whether ℓ_{priv} was visited or not.

▶ **Definition 9** (full ET-opacity)**.** *A TA $v(\mathcal{A})$ is* fully ET-opaque *if* $DVisit^{priv}(v(\mathcal{A})) = DVisit^{\overline{priv}}(v(\mathcal{A}))$.

▶ **Example 10.** Consider again the PTA $\mathcal{A}$ in Figure 1a. Let v s.t. $v(p_1) = 1$ and $v(p_2) = 4$. Then $v(\mathcal{A})$ is $\exists$-ET-opaque since there is at least one execution time for which $v(\mathcal{A})$ is ET-opaque. Here, $v(\mathcal{A})$ is ET-opaque for execution times $[1, 3]$. However, $v(\mathcal{A})$ is not fully ET-opaque since there is at least one execution time for which $v(\mathcal{A})$ is not ET-opaque. Here, $v(\mathcal{A})$ is not ET-opaque for execution times $[0, 1)$ (which can only occur on a public run) and for execution times $(3, 4]$ (which can only occur on a private run).

Let us consider the following decision problems:

$\exists$-ET-opacity p emptiness problem ($\exists$OE):
INPUT: A PTA $\mathcal{A}$
PROBLEM: Decide the emptiness of the set of valuations v s.t. $v(\mathcal{A})$ is $\exists$-ET-opaque.

Full ET-opacity p emptiness problem (FOE):
INPUT: A PTA $\mathcal{A}$
PROBLEM: Decide the emptiness of the set of valuations v s.t. $v(\mathcal{A})$ is fully ET-opaque.

The synthesis counterpart allows for a higher-level problem aiming at synthesizing (ideally the entire set of) parameter valuations v for which $v(\mathcal{A})$ is $\exists$-ET-opaque or fully ET-opaque.

$\exists$-ET-opacity p synthesis problem ($\exists$OS):
INPUT: A PTA $\mathcal{A}$
PROBLEM: Synthesize the set V of valuations s.t. $v(\mathcal{A})$ is $\exists$-ET-opaque, for all $v \in V$.

Full ET-opacity p synthesis problem (FOS):
INPUT: A PTA $\mathcal{A}$
PROBLEM: Synthesize the set V of valuations s.t. $v(\mathcal{A})$ is fully ET-opaque, for all $v \in V$.

3 A parametric execution times-based semi-algorithm for ∃OS and FOS

One of our main results is the proof that both ∃OS and FOS can be deduced from set operations on two sets representing respectively all the durations and parameter valuations of the runs for which ℓ_{priv} is reached (resp. avoided) on the way to ℓ_f. Those sets can be seen as a parametrized version of $DVisit^{priv}(v(\mathcal{A}))$ and $DVisit^{\overline{priv}}(v(\mathcal{A}))$. In order to compute such sets, we propose here the novel notion of parametric execution times. (Note that our partial solution for PET construction and semi-algorithms for ∃OS and FOS work perfectly for *rational*-valued parameters too, and that they are not restricted to 1-clock PTAs.)

3.1 Parametric execution times

The parametric execution times (PET) are the parameter valuations and execution times of the runs to ℓ_f.

▶ **Definition 11.** *Given a PTA $\mathcal{A}$ with final location ℓ_f, the* parametric execution times *of $\mathcal{A}$ are defined as $PET(\mathcal{A}) = \{(v, d) \mid \exists \rho$ in $v(\mathcal{A})$ such that $d = dur(\rho) \wedge \rho$ is of the form $(\ell_0, \mu_0), (d_0, e_0), \cdots, (\ell_n, \mu_n)$ for some $n \in \mathbb{N}$ such that $\ell_n = \ell_f$ and $\forall 0 \leq i \leq n-1, \ell_i \neq \ell_f\}$.*

By definition, we only consider paths up to the point where ℓ_f is reached, meaning that executions times do not include the time elapsed in ℓ_f, and that runs that reach ℓ_f more than once are only considered up to their first visit of ℓ_f.

▶ **Example 12.** Consider again the PTA $\mathcal{A}$ in Figure 1a. Then $PET(\mathcal{A})$ is $(d \leq 3 \wedge p_1 \geq 0 \wedge p_2 \geq 0) \vee (0 \leq p_1 \leq 3 \wedge p_1 \leq d \leq p_2)$.

3.1.1 Partial solution

Synthesizing parametric execution times is in fact equivalent to a reachability synthesis where the PTA is enriched (in particular by adding a clock measuring the total execution time).

▶ **Proposition 13.** *Let $\mathcal{A}$ be a PTA, and ℓ_f the final location of $\mathcal{A}$.*
Let $\mathcal{A}'$ be a copy of $\mathcal{A}$ s.t.:

- *a clock x_{abs} is added and initialized at 0 (it does not occur in any guard or reset);*
- *a parameter d is added;*
- *ℓ_f is made* urgent *(i.e., time is not allowed to pass in ℓ_f), all outgoing edges from ℓ_f are pruned and a guard $x_{abs} = d$ is added to all incoming edges to ℓ_f.*

Then, $PET(\mathcal{A}) = \mathsf{EFsynth}(\mathcal{A}', \{\ell_f\})$.

Proof. See Appendix B.1. ◀

▶ **Example 14.** Consider again the PTA $\mathcal{A}$ in Figure 1a. Then $\mathcal{A}'$ is given in Figure 1b.

As per Lemma 35 in Appendix A, there exist semi-algorithms for reachability synthesis, and hence for the PET synthesis problem—although they do not guarantee termination.

3.2 $\exists$OS and FOS problems

Now, we detail how the PET can be used to compute the solution to both $\exists$OS and FOS. To do so, we will go trough a (larger) intermediate problem: the synthesis of both parameter valuations v *and* execution times for which $v(\mathcal{A})$ is ET-opaque.

> **$\exists$-ET-opacity p-d synthesis problem (d-$\exists$OS):**
> INPUT: A PTA $\mathcal{A}$
> PROBLEM: Synthesize the set of parameter valuations v and executions times d s.t. $v(\mathcal{A})$ is $\exists$-ET-opaque and $v(\mathcal{A})$ is ET-opaque for execution time d.

> **Full ET-opacity p-d synthesis problem (d-FOS):**
> INPUT: A PTA $\mathcal{A}$
> PROBLEM: Synthesize the set of parameter valuations v and executions times d s.t. $v(\mathcal{A})$ is fully ET-opaque and $v(\mathcal{A})$ is ET-opaque for execution time d.

First, given a PTA $\mathcal{A}$ and two locations ℓ_f and ℓ_{priv} of $\mathcal{A}$, let us formally define both sets representing respectively all the durations and parameter valuations of the runs for which ℓ_{priv} is reached (resp. avoided) on the way to ℓ_f.

Let $\mathcal{A}_{\ell_f}^{\ell_{priv}}$ be a copy of $\mathcal{A}$ s.t.: 1) a Boolean variable[1] b is added and initialized to *False*, 2) b is set to *True* on all incoming edges to ℓ_{priv}, 3) a guard $b = \mathit{True}$ is added to all incoming edges to ℓ_f. The PTA $\mathcal{A}_{\ell_f}^{\ell_{priv}}$ contains all runs of $\mathcal{A}$ for which ℓ_{priv} is reached on the way to ℓ_f, and $PET(\mathcal{A}_{\ell_f}^{\ell_{priv}})$ contains the durations and parameter valuations of those runs.

Let $\mathcal{A}_{\ell_f}^{\neg\ell_{priv}}$ be a copy of $\mathcal{A}$ s.t. all incoming and outgoing edges to and from ℓ_{priv} are pruned. The PTA $\mathcal{A}_{\ell_f}^{\neg\ell_{priv}}$ contains all runs of $\mathcal{A}$ for which ℓ_{priv} is avoided on the way to ℓ_f, and $PET(\mathcal{A}_{\ell_f}^{\neg\ell_{priv}})$ contains the durations and parameter valuations of those runs.

▶ **Example 15.** Consider again the PTA $\mathcal{A}$ in Figure 1a. Then $\mathcal{A}_{\ell_f}^{\ell_{priv}}$ is given in Figure 1c, and $\mathcal{A}_{\ell_f}^{\neg\ell_{priv}}$ is given in Figure 1d.

▶ **Proposition 16.** *Given a PTA $\mathcal{A}$, we have: d-$\exists$OS$(\mathcal{A}) = PET(\mathcal{A}_{\ell_f}^{\ell_{priv}}) \cap PET(\mathcal{A}_{\ell_f}^{\neg\ell_{priv}})$.*

Proof. See Appendix B.2. ◀

▶ **Example 17.** Consider again the PTA $\mathcal{A}$ in Figure 1a. Then $PET(\mathcal{A}_{\ell_f}^{\ell_{priv}})$ is $p_1 \leq d \leq p_2 \wedge 0 \leq p_1 \leq 3$. Moreover, $PET(\mathcal{A}_{\ell_f}^{\neg\ell_{priv}})$ is $0 \leq d \leq 3 \wedge p_1 \geq 0 \wedge p_2 \geq 0$. Hence, d-$\exists$OS$(\mathcal{A})$ is $0 \leq p_1 \leq d \leq p_2 \wedge d \leq 3$.

In order to compute d-FOS$(\mathcal{A})$, we need to remove from d-$\exists$OS$(\mathcal{A})$ all parameter valuations v s.t. there is at least one run to ℓ_f in $v(\mathcal{A})$ whose duration is not in D_v. Parameter valuations and durations of such runs are included in $PET(\mathcal{A}) \setminus$ d-$\exists$OS$(\mathcal{A})$, which is also the difference between $PET(\mathcal{A}_{\ell_f}^{\ell_{priv}})$ and $PET(\mathcal{A}_{\ell_f}^{\neg\ell_{priv}})$. We note that difference as

$$\mathit{Diff}(\mathcal{A}) = \big(PET(\mathcal{A}_{\ell_f}^{\ell_{priv}}) \cup PET(\mathcal{A}_{\ell_f}^{\neg\ell_{priv}})\big) \setminus \big(PET(\mathcal{A}_{\ell_f}^{\ell_{priv}}) \cap PET(\mathcal{A}_{\ell_f}^{\neg\ell_{priv}})\big)$$

$\mathit{Diff}(\mathcal{A})$ is made of a union of convex polyhedra $\mathbf{C}$ over $\mathbb{P}$ (i.e., the parameters of $\mathcal{A}$) and d, which is the duration of runs. The parameter values in those polyhedra are the ones we do not want to see in d-FOS$(\mathcal{A})$. Our solution thus consists in removing from d-$\exists$OS$(\mathcal{A})$ the values of $\mathbb{P}$ in $\mathit{Diff}(\mathcal{A})$.

[1] Which is a convenient syntactic sugar for doubling the number of locations.

▶ **Proposition 18.** *Given a PTA $\mathcal{A}$ with parameter set $\mathbb{P}$:* $\texttt{d-FOS}(\mathcal{A}) = \texttt{d-}\exists\texttt{OS}(\mathcal{A}) \setminus \mathit{Diff}(\mathcal{A})\downarrow_{\mathbb{P}}$.

Proof. See Appendix B.3. ◀

▶ **Example 19.** Consider again the PTA $\mathcal{A}$ in Figure 1a. Whe have $\mathit{Diff}(\mathcal{A})$ is $(0 \leq p_1 \leq 3 < d \leq p_2) \vee (0 \leq d \leq 3 \wedge d < p_1 \wedge p_2 \geq 0) \vee (0 \leq p_2 < d \leq 3 \wedge p_1 \geq 0)$. Then $\mathit{Diff}(\mathcal{A})\downarrow_{\mathbb{P}}$ is $(0 \leq p_1 \leq 3 < p_2) \vee (0 < p_1 \wedge p_2 \geq 0) \vee (0 \leq p_2 < 3 \wedge p_1 \geq 0)$. Hence, $\texttt{d-FOS}(\mathcal{A})$ is $p_1 = 0 \leq d \leq p_2 = 3$.

Finally, obtaining $\exists\texttt{OS}(\mathcal{A})$ and $\texttt{FOS}(\mathcal{A})$ is trivial since, by definition, $\exists\texttt{OS}(\mathcal{A}) = (\texttt{d-}\exists\texttt{OS}(\mathcal{A}))\downarrow_{\mathbb{P}}$ and $\texttt{FOS}(\mathcal{A}) = (\texttt{d-FOS}(\mathcal{A}))\downarrow_{\mathbb{P}}$.

▶ **Example 20.** Consider again the PTA $\mathcal{A}$ in Figure 1a. Then $\exists\texttt{OS}(\mathcal{A})$ is $0 \leq p_1 \leq p_2 \wedge p_1 \leq 3$. And $\texttt{FOS}(\mathcal{A})$ is $p_1 = 0 \wedge p_2 = 3$.

3.2.1 On correctness and termination

We described here a method for computing $\exists\texttt{OS}(\mathcal{A})$ and $\texttt{FOS}(\mathcal{A})$ for a PTA, that produces an exact (sound and complete) result if it terminates. It relies on the PET of two subsets of the PTA, which computation requires enrichment with one clock and one parameter. If they can be computed, those PET take the form of a finite union of convex polyhedra, on which are then applied the union, intersection, difference and projection set operations—that are known to be decidable in this context. Thus the actual termination of the whole semi-algorithm relies on the reachability synthesis of two $(n+1, m+1)$-PTAs. Reachability synthesis is known to be effectively computable for $(1, m)$-PTAs [8], and cannot be achieved for PTAs with 3 parametric clocks due to the undecidability of the reachability emptiness problem [2]. For the semi-algorithm we proposed here for $\exists\texttt{OS}$ and $\texttt{FOS}$ problems, we therefore do not have any guarantees of termination, even with only one parametric clock (due to the additional clock x_{abs}), although this might change depending on future results regarding the decidability of reachability synthesis for PTAs with 2 parametric clocks (a first decidability result for the emptiness only was proved for $(2, *, 1)$-PTAs over discrete time [17]).

4 Decidability and undecidability of FOE for 1-clock-PTAs

In this section, we:

1. propose a method to compute potentially infinite PET on $(1, 0, *)$-PTAs, i.e., PTAs with 1 parametric clock and arbitrarily many parameters (Section 4.1);
2. prove decidability of the $\texttt{FOE}$ problem for $(1, 0, 1)$-PTAs, by rewriting infinite PET in a variant of Presburger arithmetic (Section 4.2);
3. prove undecidability of the $\texttt{FOE}$ problem for $(1, 0, *)$-PTAs (Section 4.2).

4.1 Encoding infinite PET for $(1, 0, *)$-PTAs

Given a PTA $\mathcal{A}$ with exactly 1 clock, the goal of the method described here is to guarantee termination of the computation of $\mathit{PET}(\mathcal{A})$ with an exact result. If applying the general method given in Section 3.1, it would amount to a reachability synthesis on a PTA with 2 clocks, without guarantee of termination. The gist of this method is a form of divide and conquer, where we solve sub-problems, specifically reachability synthesis on sub-parts of $\mathcal{A}$ without adding an additional clock. The first step consists in building some reset-free PTAs, each representing a meaningful subset of the paths joining two given locations in $\mathcal{A}$.

$PET(\mathcal{A})$ is then obtained by combining the results of reachability synthesis performed on those reset-free PTAs. The result is encoded in a (finite) regular expression that represents an infinite union of convex polyhedra. Note that this method work perfectly for rational-valued parameters.

4.1.1 Defining the set of reset-free PTAs

Each of the PTAs we build describes parts of the behavior between two locations. More precisely, they represent all the possible paths such that clock resets may occur only on the last transition of the path. We first define the set of locations that we may need based on whether they are initial, final, or reached by a transition associated to a reset.

▶ **Definition 21** (Final-reset paths $FrP(\mathcal{A}, \ell_\mathrm{f})$). *Let $\mathcal{A}$ be a 1-clock PTA, ℓ_0 its initial location and ℓ_f a location of $\mathcal{A}$. We define as $FrP(\mathcal{A}, \ell_\mathrm{f})$ the set of pairs of locations s.t. $\forall(\ell_i, \ell_j) \in FrP(\mathcal{A}, \ell_\mathrm{f})$*

- *$\ell_i = \ell_0$, or $\ell_i \neq \ell_\mathrm{f}$ and there is a clock reset on an ongoing edge to ℓ_i,*
- *$\ell_j = \ell_\mathrm{f}$, or there is a clock reset on an ongoing edge to ℓ_j.*

For each pair of states (ℓ_i, ℓ_j) as defined above, we build a reset-free PTA. If the target state ℓ_j is not final (which is a special case), the reset-free PTA models every path going from ℓ_i to ℓ_j and that ends with a reset on its last step. In particular, this ensures that ℓ_j is reached with clock valuation 0.

▶ **Definition 22** (Reset-free PTA $\mathcal{A}(\ell_i, \ell_j)$). *Let $\mathcal{A}$ be a 1-clock PTA, x its unique clock, and ℓ_i, ℓ_j two locations in $\mathcal{A}$. We define as $\mathcal{A}(\ell_i, \ell_j)$ the reset-free PTA obtained from a copy of $\mathcal{A}$ by:*

1. *creating a duplicate ℓ'_j of ℓ_j;*
2. *for all incoming edges (ℓ, g, a, R, ℓ_j) where $R \in \emptyset$, removing (ℓ, g, a, R, ℓ_j) and adding an incoming edge (ℓ, g, a, R, ℓ'_j);*
3. *if $\ell_j \neq \ell_\mathrm{f}$, then for all outgoing edges (ℓ_j, g, a, R, ℓ), removing (ℓ_j, g, a, R, ℓ) and adding an outgoing edge (ℓ'_j, g, a, R, ℓ),*
   else, making ℓ'_j urgent and adding an edge $(\ell'_j, True, \epsilon, \emptyset, \ell_j)$;
4. *removing any upper bound invariant on ℓ_j and making it urgent;*
5. *if $\ell_i \neq \ell_j$, setting ℓ_i as the initial location,*
   else, setting ℓ'_j as the initial location;
6. *removing any clock reset on incoming edges to ℓ_j and pruning all other edges featuring a clock reset, and all outgoing edges from ℓ_f;*
7. *adding a parameter d, and a guard $x = d$ to all incoming edges to ℓ_j;*

We will show next how the reachability synthesis of those reset-free PTAs corresponds to fragments of the runs that are considered in $PET(\mathcal{A})$. The following two proposition will be needed for that demonstration. For simplification, given $\mathcal{A}$ a 1-clock PTA, and ℓ_i, ℓ_j two locations of $\mathcal{A}$, we now note $Z_{\ell_i,\ell_j} = \mathsf{EFsynth}(\mathcal{A}(\ell_i, \ell_j), \{\ell_j\})$.

▶ **Proposition 23.** *Let $\mathcal{A}$ be a 1-clock PTA, and $(\ell_i, \ell_j) \in FrP(\mathcal{A}, \ell_\mathrm{f})$ such that $\ell_j \neq \ell_\mathrm{f}$. Then Z_{ℓ_i,ℓ_j} is equivalent to the synthesis of parameter valuations v and execution times D_v such that $D_v = \{d \mid \exists\rho$ from $(\ell_i, \{x = 0\})$ to ℓ_j in $v(\mathcal{A})$ such that $d = dur(\rho)$, ℓ_f is never reached, and x is reset on the last edge of ρ and on this edge only $\}$.*

Proof. See Appendix B.4. ◀

▶ **Proposition 24.** *Let $\mathcal{A}$ be a 1-clock PTA, and $(\ell_i, \ell_j) \in FrP(\mathcal{A}, \ell_\mathrm{f})$ such that $\ell_j = \ell_\mathrm{f}$. Then Z_{ℓ_i,ℓ_j} is equivalent to the synthesis of parameter valuations v and execution times D_v such that $D_v = \{d \mid \exists \rho$ from $(\ell_i, \{x = 0\})$ to ℓ_f in $v(\mathcal{A})$ such that $d = dur(\rho)$, ℓ_f is reached only on the last state of ρ, and x may only be reset on the last edge of ρ $\}$.*

Proof. See Appendix B.5. ◀

4.1.2 Reconstruction of PET from the reachability synthesis of the reset-free PTAs.

Given $\mathcal{A}$ a 1-clock PTA, and ℓ_f a location of $\mathcal{A}$, for all $(\ell_i, \ell_j) \in FrP(\mathcal{A}, \ell_\mathrm{f})$ we may compute the parametric zone Z_{ℓ_i,ℓ_j} with guarantee of termination, since the reachability synthesis is decidable on 1-clock PTAs. Those parametric zones may be used to build the (potentially infinite) PET of $\mathcal{A}$. To do so, we first define a (non-parametric, untimed) finite automaton where the states are the locations of $\mathcal{A}$, and the arc between the states ℓ_i and ℓ_j is labeled by Z_{ℓ_i,ℓ_j}. We refer to this automaton as the *automaton of the zones* of $\mathcal{A}$.

▶ **Definition 25** (Automaton of the zones)**.** *Let $\mathcal{A}$ be a 1-clock PTA, ℓ_0 its initial location and ℓ_f a location of $\mathcal{A}$. We define as $\hat{\mathcal{A}}$ the finite automaton such that:*

- *The states of $\hat{\mathcal{A}}$ are exactly the locations of $\mathcal{A}$;*
- *ℓ_0 is initial and ℓ_f is final;*
- *$\forall (\ell_i, \ell_j) \in FrP(\mathcal{A}, \ell_\mathrm{f})$, there is a transition from ℓ_i to ℓ_j labelled by Z_{ℓ_i,ℓ_j}.*

We claim that the language $\hat{L}$ of $\hat{\mathcal{A}}$ is a representation of the times (along with parameter constraints) to go from ℓ_0 to ℓ_f in $\mathcal{A}$. As $\hat{\mathcal{A}}$ is a finite automaton, $\hat{L}$ can be represented as a regular expression with three operators: the concatenation (.), the alteration (+), and the Kleene star (*). $PET(\mathcal{A})$ can thus be expressed by redefining those operators with operations on the parametric zones that label edges of $\hat{L}$.

Any parametric zone $Z_{a,b}$ labeling an edge of $\hat{\mathcal{A}}$ is of the form $\bigcup_i \mathbf{C}_i$ with $1 \leq i \leq n$ and $\mathbf{C}_i$ a convex polyhedra. As per Definition 6, $\mathbf{C}_i$ is a conjunction of inequalities, each of the form $\alpha d + \sum_{1 \leq i \leq M} \beta_i p_i + \gamma \bowtie 0$, with $p_i \in \mathbb{P}$, and $\alpha, \beta_i, \gamma \in \mathbb{Z}$. Note that x has been replaced by execution times d, as per Definition 11. In the following, we note by $\mathbf{C}_i^d$ all inequalities such that $\alpha \neq 0$ (i.e., inequalities over d and possibly some parameters in $\mathbb{P}$), and by $\mathbf{C}_i^\mathbb{P}$ all inequalities such that $\alpha = 0$ (i.e., inequalities strictly over $\mathbb{P}$). This means that $\mathbf{C}_i = \mathbf{C}_i^d \cap \mathbf{C}_i^\mathbb{P}$. For simplification of what follows, we write inequalities in $\mathbf{C}_i^d$ as $d \bowtie c$ where $c = \frac{\sum_{1 \leq i \leq M} \beta_i p_i + \gamma}{-\alpha}$.

Given $Z_{a,b} = \bigcup_i \mathbf{C}_i$ and $Z_{c,d} = \bigcup_j \mathbf{C}_j$, we define the operators $\bar{.}$, $\bar{*}$ and $\bar{+}$.

Operator $\bar{.}$ is the addition of the time durations and intersection of parameter constraints between two parametric zones. Formally, $Z_{a,b} \mathbin{\bar{.}} Z_{c,d} = \bigcup_{i*j} \mathbf{C}_{i,j}^d \cap \mathbf{C}_{i,j}^\mathbb{P}$ such that $\mathbf{C}_{i,j}^\mathbb{P} = \mathbf{C}_i^\mathbb{P} \cap \mathbf{C}_j^\mathbb{P}$, and for all $d \bowtie c_i \in \mathbf{C}_i^d$ and $d \bowtie' c_j \in \mathbf{C}_j^d$, if $\bowtie, \bowtie' \in \{<, \leq, =\}$ or $\bowtie, \bowtie' \in \{>, \geq, =\}$, then $d \bowtie'' c_i + c_j \in \mathbf{C}_{i,j}^d$ with $\bowtie''$ being in the same direction as $\bowtie$ and $\bowtie'$ and is

- a strict inequality if either $\bowtie$ or $\bowtie'$ is a strict inequality;
- a strict equality if both $\bowtie$ and $\bowtie'$ are strict equalities;
- a non-strict inequality otherwise.

Operator $\bar{*}$ is the recursive application of $\bar{.}$ on a parametric zone. Formally, $Z_{a,b}{}^{\bar{*}} = \bigcup_{K \in \mathbb{N}} \{d = 0\}(\bar{.} Z_{a,b})^K$ where $(\bar{.} Z_{a,b})$ is repeated K times, with K being any value in $\mathbb{N}$. Note that $\{d = 0\}$ corresponds to the case where the loop is never taken, and that it is neutral for the $\bar{\cdot}$ operator: $\{d = 0\}\bar{\cdot}Z_{a,b} = Z_{a,b}$. Also note that, in practice, $a = b$ whenever we use this operator.

Operator $\bar{+}$ is the union of two parametric zones. Formally, $Z_{a,b}\bar{+}Z_{c,d} = Z_{a,b} \cup Z_{c,d}$.

Note that the result of any of those operations is a union of convex polyhedra of the form $\bigcup_i \mathbf{C}_i$, meaning that these operators can be nested. Also, this union is infinite whenever operator $\bar{*}$ is present.

▶ **Proposition 26.** *Let $\mathcal{A}$ be a 1-clock PTA and ℓ_{f} a location of $\mathcal{A}$. Let $\hat{L}$ be the language of the automaton of the zones $\hat{\mathcal{A}}$, and e a regular expression describing $\hat{L}$. Let $\bar{e}$ be the expression obtained by replacing the $.$, $+$ and * operators in e respectively by $\bar{\cdot}$, $\bar{+}$ and $\bar{*}$. We have $\bar{e} = PET(\mathcal{A})$.*

Proof. See Appendix B.6. ◀

4.2 Solving the FOE problem through a translation of PET to parametric Presburger arithmetic

Presburger arithmetic is the first order theory of the integers with addition. It is a useful tool that can represent and manipulate sets of integers called semi-linear sets. Those sets are particularly meaningful to study TAs, as the set of durations of runs reaching the final location can be described by a semi-linear set [11]. Presburger arithmetic is however not expressive enough to represent durations of runs in PTAs due to the presence of parameters. In [22], a parametric extension of Presburger arithmetic was considered, introducing linear parametric semi-linear sets (LpSl sets) which are functions associating to a parameter valuation v a (traditional) semi-linear set of the following form:

$$S(v) = \Big\{x \in \mathbb{N}^m \mid \bigvee_{i \in I} \exists x_0, \dots x_{n_i} \in \mathbb{N}^m, k_1, \dots k_{n_i} \in \mathbb{N}, x = \sum_{j=0}^{n_i} x_j \\ \wedge b_0^i(v) \leq x_0 \leq c_0^i(v) \wedge \bigwedge_{j=1}^{n_i} k_j b_j^i(v) \leq x_j \leq k_j c_j^i(v)\Big\} \tag{1}$$

where I is a finite set and the b_j^i and c_j^i are linear polynomials with coefficients in $\mathbb{N}$. A 1-LpSl set is an LpSl set defined over a single parameter. Given two LpSl (resp. 1-LpSl) sets S_1 and S_2, the LpSl (resp. 1-LpSl) equality problem consists in deciding whether there exists a parameter valuation v such that $S_1(v) = S_2(v)$.

▶ **Theorem 27** ([22]). *The LpSl equality problem is undecidable.*

The 1-LpSl equality problem is decidable. Moreover, the set of valuations achieving equality can be computed.

The main goal of this subsection is to relate the expressions computed in Section 4.1 to LpSl sets in order to tackle ET-opacity problems. Since Presburger arithmetic is a theory of integers, we have to restrict PTAs to integer parameters; this is what prevents our results to be extended to rational-valued parameters in a straightforward manner. Moreover, we need to focus on time durations of runs with integer values. This second restriction however is without loss of generality. Indeed, in [6, Theorem 5], a trick is provided (which consists mainly in doubling every term of the system so that any run duration that used to be a rational of the form $\frac{q}{2}$ is now an integer to ensure that if a set is non-empty, it contains an integer. This transformation also allows one to consider only non-strict constraints, and thus we assume every constraint is non-strict in the following.

▶ **Theorem 28.** *The LpSl equality problem reduces to the* `FOE` *problem for* $(1, 0, *)$*-PTAs. Moreover, the* `FOE` *problem for* $(1, 0, 1)$*-PTAs reduces to the 1-LpSl equality problem.*

Sketch of proof. From Equation (1) one can see that an LpSl set parametrically defines integers that are the sum of two types of elements: x_0 belongs to an interval, while the x_j represent a sum of integers, each coming from the interval $[b^i_j; c^i_j]$. Intuitively, we separate a run into its elementary path until the final state and its loops. We use x_0 to represent the duration of the elementary path, and the x_j adds the duration of loops. Each occurrence of the same loop within a run being independent (as they include a reset of the clock), their durations all belong to the same interval.

Formally, given a PTA $\mathcal{A}$, using Section 3.2, we build the PTAs $\mathcal{A}^{\ell_{priv}}_{\ell_f}$ and $\mathcal{A}^{\neg\ell_{priv}}_{\ell_f}$ separating the private and public runs of $\mathcal{A}$. Then with Section 4.1, we obtain expressions $\bar{e}_{\ell_{priv}}$ and $\bar{e}_{\neg\ell_{priv}}$ such that (Proposition 26) $\bar{e}_{\ell_{priv}} = PET(\mathcal{A}^{\ell_{priv}}_{\ell_f})$ and $\bar{e}_{\neg\ell_{priv}} = PET(\mathcal{A}^{\neg\ell_{priv}}_{\ell_f})$. We then develop and simplify these expressions until we can build LpSl sets representing the integers accepted by each expression. We can then show the inter-reduction as the full ET-opacity is directly equivalent to the equality of the two sets. Note that one direction of the reduction is stronger, allowing multiple parameters. This is due to constraints over the parameters which may appear in our expressions, but cannot be transferred to LpSl sets. However, when there is a single parameter, one can easily resolve these constraints beforehand. ◀

Combining Theorems 27 and 28 directly gives us:

▶ **Corollary 29.** `FOE` *is undecidable for* $(1, 0, *)$*-PTAs.*

▶ **Corollary 30.** `FOE` *is decidable for* $(1, 0, 1)$*-PTAs and* `FOS` *can be solved.*

5 Decidability of ∃`OE` for $(1, 0, *)$-PTAs for integer-valued parameters

We prove here the decidability of ∃`OE` for $(1, 0, *)$-PTAs with integer parameters over dense time (Section 5.1); we also prove that the same problem is in EXPSPACE for $(1, *, 1)$-PTAs over discrete time (Section 5.2).

5.1 General case

Adding the divisibility predicate (denoted "|") to Presburger arithmetic produces an undecidable theory, whose purely existential fragment is known to be decidable [21]. The `FOE` problem can be encoded in this logic, but requires a single quantifier alternation, which goes beyond the aforementioned decidability result, leading us to rely on [22]. The ∃`OE` problem however can be encoded in the purely existential fragment.

▶ **Theorem 31.** *The* ∃`OE` *problem is decidable.*

Sketch of proof. As for Theorem 28, we start by building and simplifying expressions representing the private and public durations of the PTA. Instead of translating the expression into LpSl set however, we now use Presburger with divisibility.

Again, a run can be decomposed in the run without loop and its loops. The duration of the former is defined directly by conjunction of inequalities, which can be formulated in a Presburger arithmetic formula. The latter requires the divisibility operator to represent the arbitrary number of loops. Hence, we can build a formula accepting exactly the integers satisfying our expressions. Deciding the ∃`OE` problem can be achieved by testing the existence of an integer satisfying the formulas produced from both expression, which can be stated in a purely existential formula. ◀

Table 1 Execution-time opacity problems for PTAs: contributions and some open cases

Time	(pc, npc, p)	$\exists$`OE` emptiness	$\exists$`OE` synthesis
dense	$(1, 0, *)$	$\surd$ (Th. 31)	?
dense	$(1, *, *)$	?	?
dense	$(2, 0, 1)$	?	?
dense	$(3, 0, 1)$	$\times$ ([7, Th.6.1])	$\times$
discrete	$(1, *, 1)$	$\surd$EXPSPACE (Th. 33)	?

Time	(pc, npc, p)	`FOE` emptiness	`FOE` synthesis
dense	$(1, 0, 1)$	$\surd$ Corol. 30	$\surd$ Corol. 30
dense	$(1, 0, [2, M])$	?	?
dense	$(1, 0, M)$	$\times$ (Corol. 29)	$\times$
dense	$([2, 3], 0, 1)$	?	?
dense	$(4, 0, 2)$	$\times$ ([7, Th. 7.1])	$\times$

▶ Remark 32 (complexity). Let us quickly discuss the complexity of this algorithm. The expressions produced by Proposition 26 can, in the worst case, be exponential in the size of the PTA. This formula was then simplified within the proof of Theorem 28, in part by developing it, which could lead to an exponential blow-up. Finally, the existential fragment of Presburger arithmetic with divisibility can be solved in NEXPTIME [21]. As a consequence, our algorithm lies in 3NEXPTIME.

5.2 Discrete time case

There are clear ways to improve the complexity of this algorithm. In particular, we finally prove an alternative version of Theorem 31 in a more restricted setting ($\mathbb{T} = \mathbb{N}$), but with a significantly lower complexity upper bound and using completely different proof ingredients.

▶ **Theorem 33.** *$\exists$`OE` is decidable in* EXPSPACE *for* $(1, *, 1)$*-PTAs over discrete time.*

▶ Remark 34. The fact that we can handle arbitrarily many non-parametric clocks in Theorem 33 does not improve Theorem 31: over discrete time, it is well-known that non-parametric clocks can be eliminated using a technique from [2], and hence come "for free".

6 Conclusion and perspectives

In this paper, we addressed the ET-opacity for 1-clock PTAs with integer-valued parameters over dense time. We proved that 1) `FOE` is undecidable for a sufficiently large number of parameters, 2) `FOE` becomes decidable for a single parameter, and 3) $\exists$`OE` is decidable, in 3NEXPTIME over dense time and in EXPSPACE over discrete time. These results rely on a novel construction of PET, for which a sound and complete computation method is provided. In the general case, we provided semi-algorithms for the computation of PET, $\exists$`OS` and `FOS`.

Our PET constructions and all PET-related results work perfectly for rational-valued parameters. It remains however unclear how to extend our (un)decidability results to rational-valued parameters, as our other proof ingredients (notably using the Presburger arithmetics) heavily rely on integer-valued parameters.

It remains also unclear whether synthesis can be achieved using techniques from [17], explaining the "open" cell in the "discrete time" row of Table 1. Also, a number of problems remain open in Table 1, notably the 2-clock case, already notoriously difficult for reachability emptiness [2, 17].

Finally, exploring *weak* ET-opacity [5] is also on our agenda.

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A Recalling the correctness of EFsynth

▶ **Lemma 35** ([20]). *Let $\mathcal{A}$ be a PTA, and let L_{target} be a subset of the locations of $\mathcal{A}$. Assume $\mathsf{EFsynth}(\mathcal{A}, L_{target})$ terminates with result K. Then $v \models K$ iff L_{target} is reachable in $v(\mathcal{A})$.*

B Proof of results

B.1 Proof of Proposition 13

▶ **Proposition 13.** *Let $\mathcal{A}$ be a PTA, and ℓ_f the final location of $\mathcal{A}$.*
Let $\mathcal{A}'$ be a copy of $\mathcal{A}$ s.t.:

- *a clock x_{abs} is added and initialized at 0 (it does not occur in any guard or reset);*
- *a parameter d is added;*
- *ℓ_f is made* urgent *(i.e., time is not allowed to pass in ℓ_f), all outgoing edges from ℓ_f are pruned and a guard $x_{abs} = d$ is added to all incoming edges to ℓ_f.*

Then, $PET(\mathcal{A}) = \mathsf{EFsynth}(\mathcal{A}', \{\ell_f\})$.

Proof. By having ℓ_f being urgent and removing its outgoing edges, we ensure that the runs that reach ℓ_f in $\mathcal{A}'$ are all of the form $(\ell_0, \mu_0), (d_0, e_0), \cdots, (\ell_n, \mu_n)$ for some $n \in \mathbb{N}$ such that $\ell_n = \ell'$ and $\forall 0 \leq i \leq n-1, \ell_i \neq \ell'$. By having a clock x_{abs} that is never reset and ℓ_f being urgent, we ensure that for any run ρ that reaches ℓ_f in $\mathcal{A}'$, the value of x_{abs} in the final state if equals to $dur(\rho)$. By having a guard $x_{abs} = d$ on all incoming edges to ℓ_f, we ensure that $d = dur(\rho)$ on any run ρ that reaches ℓ_f.

Therefore, $\mathsf{EFsynth}(\mathcal{A}', \{\ell_f\})$ contains all parameter valuations of the runs to ℓ_f in $\mathcal{A}$ that stop once ℓ_f is reached, along with the duration of those runs contained in d. ◀

B.2 Proof of Proposition 16

▶ **Proposition 16.** *Given a PTA $\mathcal{A}$, we have: $\mathtt{d}\text{-}\exists\mathit{OS}(\mathcal{A}) = PET(\mathcal{A}_{\ell_f}^{\ell_{priv}}) \cap PET(\mathcal{A}_{\ell_f}^{\neg\ell_{priv}})$.*

Proof. By definition, $\texttt{d-}\exists\texttt{OS}(\mathcal{A})$ is the synthesis of parameter valuations v and execution times D_v such that $v(\mathcal{A})$ is opaque w.r.t. ℓ_{priv} on the way to ℓ_f for these execution times D_v. This means that $\texttt{d-}\exists\texttt{OS}(\mathcal{A})$ contains exactly all parameter valuations and executions times for which there exist both at least one run in $\mathcal{A}_{\ell_\mathrm{f}}^{\ell_{priv}}$ and at least one run in $\mathcal{A}_{\ell_\mathrm{f}}^{\ell_{priv}}$. Since PET are the synthesis of the parameter valuations and execution times up to the final location, $\texttt{d-}\exists\texttt{OS}(\mathcal{A})$ is equivalent to the intersection of the $PET(\mathcal{A}_{\ell_\mathrm{f}}^{\ell_{priv}})$ and $PET(\mathcal{A}_{\ell_\mathrm{f}}^{\neg\ell_{priv}})$. ◀

B.3 Proof of Proposition 18

▶ **Proposition 18.** *Given a PTA $\mathcal{A}$ with parameter set $\mathbb{P}$: $\texttt{d-FOS}(\mathcal{A}) = \texttt{d-}\exists\texttt{OS}(\mathcal{A}) \setminus \mathit{Diff}(\mathcal{A}){\downarrow_\mathbb{P}}$.*

Proof. By definition, $\texttt{d-FOS}(\mathcal{A})$ is the synthesis of parameter valuations v (and execution times of their runs) s.t. $v(\mathcal{A})$ is fully opaque w.r.t. ℓ_{priv} on the way to ℓ_f. By definition, $\mathit{Diff}(\mathcal{A}){\downarrow_\mathbb{P}}$ is the set of parameter valuations s.t. for any valuation $v \in \mathit{Diff}(\mathcal{A}){\downarrow_\mathbb{P}}$, there is at least one run where ℓ_{priv} is reached (resp. avoided) on the way to ℓ_f in $v(\mathcal{A})$ whose duration time is different from those of any run where ℓ_{priv} is avoided (resp. reached) on the way to ℓ_f in $v(\mathcal{A})$. By removing this set of parameters from $\texttt{d-}\exists\texttt{OS}(\mathcal{A})$, we are left with parameter valuations (and execution times of their runs) s.t. for any v, any run ρ where ℓ_{priv} is reached (resp. avoided) on the way to ℓ_f in $v(\mathcal{A})$, there is a run ρ' where ℓ_{priv} is avoided (resp. reached) on the way to ℓ_f in $v(\mathcal{A})$ and $dur(\rho) = dur(\rho')$. This is equivalent to our definition of full opacity. ◀

B.4 Proof of Proposition 23

▶ **Proposition 23.** *Let $\mathcal{A}$ be a 1-clock PTA, and $(\ell_i, \ell_j) \in \mathit{FrP}(\mathcal{A}, \ell_\mathrm{f})$ such that $\ell_j \neq \ell_\mathrm{f}$. Then Z_{ℓ_i,ℓ_j} is equivalent to the synthesis of parameter valuations v and execution times D_v such that $D_v = \{d \mid \exists\rho \text{ from } (\ell_i, \{x = 0\}) \text{ to } \ell_j \text{ in } v(\mathcal{A}) \text{ such that } d = dur(\rho), \ell_\mathrm{f} \text{ is never reached, and } x \text{ is reset on the last edge of } \rho \text{ and on this edge only }\}$.*

Proof. Let us first consider the case where $\ell_i \neq \ell_j$. Steps 1 to 3 in Definition 22 imply that whenever ℓ_j occurs either as a source or target location in an edge, it is replaced by the duplicate locality ℓ'_j, except when ℓ_j is the target location and x is reset on the edge. At this stage, for any path between ℓ_i and ℓ_j in $\mathcal{A}$, where no incoming edge to ℓ_j featuring a clock reset is present, there is an equivalent path in $\mathcal{A}(\ell_i, \ell_j)$ with ℓ_j being replaced by ℓ'_j. Step 4 implies that whenever ℓ_j is reached in $\mathcal{A}(\ell_i, \ell_j)$ no delay is allowed. As there are no outgoings edges from ℓ_j anymore, and only incoming edges featuring a clock reset, only runs ending with such edges are accepted by the reachability synthesis on ℓ_j. Since the clock value when entering in ℓ_j through such an edge is always 0, removing the upper bound of the invariant does not impact the availability of transitions. Because of our assumption that $\ell_i \neq \ell_j$, Step 5 does not change the initial location. Step 6 ensures that, in any run from ℓ_i to ℓ_j :

- no clock reset is performed before the last edge of the run;
- the clock is not reset when entering ℓ_j, and is therefore equals to the duration of the run;
- ℓ_f is not reached.

Step 7 ensures that d is equal to the value of the clock when entering ℓ_f.

Let us now consider the case where $\ell_i = \ell_j$. In this case, Step 5 changes the initial locality to ℓ'_j. Because of Steps 1 to 3, runs from ℓ'_j to ℓ_j in $\mathcal{A}(\ell_i, \ell_j)$ are identical to runs looping from ℓ_i to ℓ_i in $\mathcal{A}$ where x is reset on the last edge of the run and on this edge only. Restrictions obtained by Steps 4, 6 and 7 are unchanged.

Therefore, Z_{ℓ_i,ℓ_j} is equivalent to the synthesis of parameter valuations v and execution times D_v such that $D_v = \{d \mid \exists\rho$ from $(\ell_i, \{x = 0\})$ to ℓ_j in $v(\mathcal{A})$ such that $d = dur(\rho)$, ℓ_f is never reached, and x is reset on the last edge of ρ and on this edge only. ◀

B.5 Proof of Proposition 24

▶ **Proposition 24.** *Let $\mathcal{A}$ be a 1-clock PTA, and $(\ell_i, \ell_j) \in FrP(\mathcal{A}, \ell_\mathrm{f})$ such that $\ell_j = \ell_\mathrm{f}$. Then Z_{ℓ_i,ℓ_j} is equivalent to the synthesis of parameter valuations v and execution times D_v such that $D_v = \{d \mid \exists\rho$ from $(\ell_i, \{x = 0\})$ to ℓ_f in $v(\mathcal{A})$ such that $d = dur(\rho)$, ℓ_f is reached only on the last state of ρ, and x may only be reset on the last edge of ρ $\}$.*

Proof. By Definition 21, we know that $\ell_i \neq \ell_\mathrm{f}$.

Steps 1 to 3 in Definition 22 imply that:

- whenever ℓ_f is the target location of an edge, it is replaced by the duplicate locality ℓ'_j, except when x is reset on the edge;
- once ℓ'_j is reached, no delay is allowed and the only available transition consists in reaching ℓ_f through an empty action ϵ.

At this stage, the only difference between path from ℓ_i to ℓ_f in $\mathcal{A}(\ell_i, \ell_j)$ and $\mathcal{A}$ is that incoming edges to ℓ_f where x is not reset now leads to ℓ'_j, and then to ℓ_f without any added elapsed time. Step 4 implies that whenever ℓ_f is reached in $\mathcal{A}(\ell_i, \ell_j)$ no delay is allowed. As ℓ_f is either entered by the immediate transition from ℓ'_j or feature a clock reset, removing the upper bound of the invariant does not impact the availability of transitions. As $\ell_i \neq \ell_\mathrm{f}$, Step 5 does not change the initial location. Step 6 ensures that, in any run from ℓ_i to ℓ_j :

- no clock reset is performed before the last edge of the run (not counting the ϵ edge from ℓ'_j to ℓ_f);
- the clock value is not reset when entering ℓ_f, and is therefore equals to the duration of the run;
- no action can be taken after reaching ℓ_f.

Step 7 ensures that d is equal to the value of the clock when entering ℓ_f.

Therefore, Z_{ℓ_i,ℓ_j} is equivalent to the synthesis of parameter valuations v and execution times D_v such that $D_v = \{d \mid \exists\rho$ from $(\ell_i, \{x = 0\})$ to ℓ_f in $v(\mathcal{A})$ such that $d = dur(\rho)$, ℓ_f is reached only on the last state of ρ, and x may only be reset on the last edge of ρ.

◀

B.6 Proof of Proposition 26

▶ **Proposition 26.** *Let $\mathcal{A}$ be a 1-clock PTA and ℓ_f a location of $\mathcal{A}$. Let $\hat{L}$ be the language of the automaton of the zones $\hat{\mathcal{A}}$, and e a regular expression describing $\hat{L}$. Let $\bar{e}$ be the expression obtained by replacing the ., + and * operators in e respectively by $\bar{.}$, $\bar{+}$ and $\bar{*}$. We have $\bar{e} = PET(\mathcal{A})$.*

Proof. Let us first show that $\bar{e}$ contains $PET(\mathcal{A})$. Let ρ be a path whose time duration and parameter constraints are in $PET(\mathcal{A})$. By definition, ρ starts at time 0 in the initial locality and ends in ℓ_f, with only one occurrence of ℓ_f in the whole path. Let us consider that the clock is reset n times before the last transition, then ρ can be decomposed as $\rho_0 \dots \rho_n$ such that:

- $\forall\, 0 \leq i < n$, sub-path ρ_i starts in ℓ_i at time valuation 0, ends in ℓ_{i+1}, contains a single reset positioned on the last transition (thus ending with time valuation 0) and does not contain any occurrence of ℓ_f;

- sub-path ρ_n starts in ℓ_n at time valuation 0, ends in ℓ_{f}, may only contain a reset on its last transition, and contains exactly one occurrence of ℓ_{f}.

By Definition 21, $\forall\, 0 \le i < n$, $(\ell_i, \ell_{i+1}) \in \mathit{FrP}(\mathcal{A}, \ell_{\mathrm{f}})$ and by Proposition 23, $Z_{\ell_i,\ell_{i+1}}$ is the synthesis of parameter valuations and execution times of that sub-path. By Definition 21, $(\ell_n, \ell_{\mathrm{f}}) \in \mathit{FrP}(\mathcal{A}, \ell_{\mathrm{f}})$ and by Proposition 24, $Z_{\ell_n,\ell_{\mathrm{f}}}$ is the synthesis of parameter and valuation times of that sub-path. By Definition 25, there is a sequence of transitions $Z_{\ell_0,\ell_1}, \dots, Z_{\ell_i,\ell_{i+1}}, \dots, Z_{\ell_n,\ell_{\mathrm{f}}}$ in the automaton of the zones $\hat{\mathcal{A}}$. By application of operators $\bar{+}$ and $\bar{*}$, that sequence thus exists in $\bar{e}$ as $Z_{\ell_0,\ell_1} \bar{\cdot} \dots \bar{\cdot} Z_{\ell_i,\ell_{i+1}} \bar{\cdot} \dots \bar{\cdot} Z_{\ell_n,\ell_{\mathrm{f}}}$. By definition of operator $\bar{\cdot}$, this expression is the intersection of all parameter constraints and the addition of all valuation times, which is equivalent to $\mathit{PET}(\mathcal{A})$.

Let us now show that $\mathit{PET}(\mathcal{A})$ contains $\bar{e}$. By application of operators $\bar{+}$ and $\bar{*}$, any word in $\bar{e}$ can be expressed as a sequence of concatenation operations $\bar{\cdot}$. By Definition 25, given a word $Z_{\ell_0,\ell_1} \bar{\cdot} \dots \bar{\cdot} Z_{\ell_i,\ell_{i+1}} \bar{\cdot} \dots \bar{\cdot} Z_{\ell_n,\ell n+1} \in \bar{e}$, we know that ℓ_0 is the initial location of $\mathcal{A}$, $\ell_{n+1} = \ell_{\mathrm{f}}$ and $\forall\, 0 \le i \le n, \ell_i \ne \ell_{\mathrm{f}}$. By Proposition 23, $\forall\, 0 \le i < n$, $Z_{\ell_i,\ell_{i+1}}$ is the synthesis of parameter valuations and execution times of paths between ℓ_i and ℓ_{i+1} in $\mathcal{A}$ such that ℓ_{f} is never reached, and x is reset on the last edge of the path and on this edge only. And by Proposition 24, $Z_{\ell_n,\ell_{\mathrm{f}}}$ is the synthesis of parameter valuations and execution times of paths between ℓ_n and ℓ_{f} in $\mathcal{A}$ such that ℓ_{f} is reached only on the last state of ρ, and x may only be reset on the last edge of ρ.

Let us assume there exists a path ρ whose time duration and parameter constraints are in $\mathit{PET}(\mathcal{A})$ such that $\rho = \rho_0 \dots \rho_n$ and:

- $\forall\, 0 \le i < n$, sub-path ρ_i starts in ℓ_i at time valuation 0, ends in ℓ_{i+1}, contains a single reset positioned on the last transition (thus ending with time valuation 0) and does not contain any occurrence of ℓ_{f};
- sub-path ρ_n starts in ℓ_n at time valuation 0, ends in ℓ_{f}, may only contain a reset on its last transition, and contains exactly one occurrence of ℓ_{f}.

Then $Z_{\ell_0,\ell_1} \bar{\cdot} \dots \bar{\cdot} Z_{\ell_i,\ell_{i+1}} \bar{\cdot} \dots \bar{\cdot} Z_{\ell_n,\ell n+1} \in \mathit{PET}(\mathcal{A})$. On the other hand, if there does not exists such a path, then there exist $0 \le i \le n$ such that $Z_{\ell_i,\ell_{i+1}} = \emptyset$. By recursive applications of operator $\bar{\cdot}$, the whole sequence is evaluated as $\emptyset$ and thus contained in $\mathit{PET}(\mathcal{A})$.

◀

B.7 Proof of Theorem 28

▶ **Theorem 28.** *The LpSl equality problem reduces to the* FOE *problem for* $(1, 0, *)$*-PTAs. Moreover, the* FOE *problem for* $(1, 0, 1)$*-PTAs reduces to the 1-LpSl equality problem.*

Proof. Given a PTA $\mathcal{A}$, we showed in Section 3.2 how to compute two PTAs $\mathcal{A}^{\ell_{priv}}_{\ell_{\mathrm{f}}}$ and $\mathcal{A}^{\neg\ell_{priv}}_{\ell_{\mathrm{f}}}$ separating the private and public runs of $\mathcal{A}$. Then in Section 4.1, we showed how to build expressions $\bar{e}_{\ell_{priv}}$ and $\bar{e}_{\neg\ell_{priv}}$ such that (Proposition 26) $\bar{e}_{\ell_{priv}} = \mathit{PET}(\mathcal{A}^{\ell_{priv}}_{\ell_{\mathrm{f}}})$ and $\bar{e}_{\neg\ell_{priv}} = \mathit{PET}(\mathcal{A}^{\neg\ell_{priv}}_{\ell_{\mathrm{f}}})$.

Remark that the operators $\bar{\cdot}$, $\bar{*}$ and $\bar{+}$ are associative and commutative; moreover, each term Z occurring in the expressions $\bar{e}_{\ell_{priv}}$ and $\bar{e}_{\neg\ell_{priv}}$ is a union of constraints $Z = \bigcup_{i'} \mathbf{C}_{i'} = \bar{\sum}_{i'} \mathbf{C}_{i'}$. As a consequence, we can thus develop the entire expression to the form

$$\bar{\sum}_{i} \, (\mathbf{C}^i_1 \bar{\cdot} \mathbf{C}^i_2 \bar{\cdot} \cdots \bar{\cdot} \mathbf{C}^i_{n_i}) \bar{\cdot} (\mathbf{C}^i_{n_i+1})^{\bar{*}} \bar{\cdot} (\mathbf{C}^i_{n_i+2})^{\bar{*}} \bar{\cdot} \cdots \bar{\cdot} (\mathbf{C}^i_{n_i+m_i})^{\bar{*}}.$$

where we put all $\bar{+}$ outside of the expression. For example, the expression $Z_1 \bar{\cdot} (Z_2)^{\bar{*}}$ where $Z_1 = \mathbf{C}_1 \cup \mathbf{C}_2$ and $Z_2 = \mathbf{C}_3 \cup \mathbf{C}_4$ is developed into $\mathbf{C}_1 \bar{\cdot} (\mathbf{C}_3)^{\bar{*}} \bar{\cdot} (\mathbf{C}_4)^{\bar{*}} \bar{+} \mathbf{C}_2 \bar{\cdot} (\mathbf{C}_3)^{\bar{*}} \bar{\cdot} (\mathbf{C}_4)^{\bar{*}}$.

As $\mathbf{C}^{\bar{*}} = \{d = 0\} \bar{+} \mathbf{C} \bar{\cdot} \mathbf{C}^{\bar{*}}$, for each $\mathbf{C}^i_{n_i+j}$ we can w.l.o.g. express term i as the union of two terms: one where $(\mathbf{C}^i_{n_i+j})^{\bar{*}}$ is removed (i.e., this loop is never taken), and one where $\mathbf{C}^i_{n_i+j}$ is concatenated to the term (i.e., the loop is taken at least once). This means that each term, is turned into 2^{m_i} terms, where we can assume w.l.o.g. that for each $j > 0$, $\mathbf{C}^i_{n_i+j} = \mathbf{C}^i_j$.

Given an expression of the above form, by definition of $\bar{\cdot}$, the product $\mathbf{C}^i_1 \bar{\cdot} \mathbf{C}^i_2 \bar{\cdot} \cdots \bar{\cdot} \mathbf{C}^i_{n_i}$ is also a conjunction of inequalities and thus can be expressed as $\mathbf{C}^d_i \cap \mathbf{C}^{\mathbb{P}}_i$ where $\mathbf{C}^{\mathbb{P}}_i$ is obtained by the constraints that do not involve d while $\mathbf{C}^d_i$ contains the constraints that involve d and potentially some parameters in $\mathbb{P}$. Note also that by the assumption that for each $j > 0$, $\mathbf{C}^i_{n_i+j} = \mathbf{C}^i_j$, any constraint that does not involve d can be removed from $\mathbf{C}^i_{n_i+j}$ without modifying the set. Therefore, the expression can now be rewritten as

$$\mathop{\overline{\Large +}}_i (\mathbf{C}^d_i \cap \mathbf{C}^{\mathbb{P}}_i) \bar{\cdot} (\mathbf{C}^i_1)^{\bar{*}} \bar{\cdot} (\mathbf{C}^i_2)^{\bar{*}} \bar{\cdot} \cdots \bar{\cdot} (\mathbf{C}^i_{m_i})^{\bar{*}}.$$

where every inequality in $\mathbf{C}^i_j$ involves d.

- Assume the expressions involve a single parameter p. Let us show that the `FOE` problem for PTAs over a single parameter reduces to the 1-LpSl equality problem.
  Every constraint on p is of the form $p \bowtie c$ with $c \in \mathbb{N}$ and $\bowtie \in \{\leq, \geq\}$. Therefore, there exists a constant M such that for all i, either the constraint $\mathbf{C}^{\mathbb{P}}_i$ is satisfied for all $p \geq M$, or it is satisfied by none.
  For any fixed valuation v, full ET-opacity of $v(\mathcal{A})$ is decidable by [5]. We thus assume that we consider only valuations of p greater than M. This can be represented by replacing every occurrence of p in the expressions by $M + p$. This can be done without loss of generality as we can independently test whether the PTA is fully ET-opaque for the finitely many integer values of p smaller than M. When solving the `FOS` problem, we thus need to include the valuations of p smaller than M that achieved equality to the valuations provided by the reduction.
  The terms $\mathbf{C}^{\mathbb{P}}_i$ being either always or never valid, one can either remove this constraint from the expression, or the term containing it producing an expression of the form

  $$\mathop{\overline{\Large +}}_i \mathbf{C}^i_0 \bar{\cdot} (\mathbf{C}^i_1)^{\bar{*}} \bar{\cdot} (\mathbf{C}^i_2)^{\bar{*}} \bar{\cdot} \cdots \bar{\cdot} (\mathbf{C}^i_{m_i})^{\bar{*}}.$$

  where every constraint involves x.
  Once again, assuming p is large enough, the constraint $\mathbf{C}^i_j$ can be assumed to be of the form $\alpha^i_j p + \beta^i_j \leq x \leq \gamma^i_j p + \delta^i_j$ where $\alpha^i_j, \beta^i_j, \gamma^i_j, \delta^i_j \in \mathbb{N}$.
  For both expressions $\bar{e}_{\ell_{priv}}$ and $\bar{e}_{\neg\ell_{priv}}$, now in the simplified form described above, we build the 1-LpSl sets $S_{\bar{e}_{\ell_{priv}}}$ and $S_{\bar{e}_{\neg\ell_{priv}}}$ where, taking the notations from Equation (1), I is the set $\overline{\Large +}$ ranges over, for $0 \leq j \leq m_i, b^i_j = \alpha^i_j p + \beta^i_j$ and $c^i_j = \gamma^i_j p + \delta^i_j$.
  For a valuation v of p, we have that $S_{\bar{e}_{\ell_{priv}}}(v)$ contains exactly the integers that satisfy $v(\bar{e}_{\ell_{priv}})$ (and similarly for $S_{\bar{e}_{\neg\ell_{priv}}}(v)$ and $v(\bar{e}_{\neg\ell_{priv}})$). Therefore, there exists a valuation such that $\mathcal{A}$ if fully opaque w.r.t. ℓ_{priv} on the way to ℓ_{f} iff there exists a parameter valuation v such that $S_{\bar{e}_{\ell_{priv}}}(v) = S_{\bar{e}_{\neg\ell_{priv}}}(v)$, establishing the reduction.
- We now wish to show that the LpSl equality problem reduces to the `FOE` problem.
  To do so, we fix two LpSl sets S_1 and S_2, we will build two automata $\mathcal{A}_1$ and $\mathcal{A}_2$ such that, for $i \in \{1, 2\}$, similarly to the previous reduction, we have that for all valuation v, $S_i(v)$ contains exactly the integers that satisfy $v(PET(\mathcal{A}_i))$.
  Let us focus on S_1 and assume it is of the form given by Equation (1). We build $\mathcal{A}_1$ so that from the initial location ℓ_0 it can take multiple transitions (one for each $i \in I$), the

ith transition being allowed if the clock lies between b_0^i and c_0^i, reset the clock and reach a state ℓ_i. From ℓ_i, there are n_i loops, and the jth loop can be taken if the clock lies between b_j^i and c_j^i and resets the clock. Moreover, a transition can be taken from ℓ_i to ℓ_f if $x = 0$.

Formally, $\mathcal{A}_1 = (\Sigma, L, \ell_0, \mathbb{X}, \mathbb{P}, I, E)$ where $\Sigma = \{\epsilon\}$, $L = \{\ell_0, \ell_\mathrm{f}\} \cup \{\ell_i \mid i \in I\}$, $\mathbb{X} = \{x\}$, $\mathbb{P}$ is the set of parameters appearing in S_1, I does not restrict the PTA (i.e., it associates $\mathbb{R}_{\geq 0}$ to every location), and finally

$$\begin{aligned} E = & \big\{(\ell_0, (b_0^i \leq x \leq c_0^i), \epsilon, \{x\}, \ell_i \mid i \in I\big\} \\ & \cup \big\{(\ell_i, (b_j^i \leq x \leq c_j^i), \epsilon, \{x\}, \ell_i \mid i \in I, 1 \leq j \leq n_i\big\} \\ & \cup \big\{(\ell_i, (x = 0), \epsilon, \emptyset, \ell_\mathrm{f} \mid i \in I\big\}. \end{aligned}$$

Thus, a run reaching ℓ_f can be decomposed into final-reset paths. In other words, there is a run reaching ℓ_f with duration d iff d can be written as a sum $d = \sum_{j=0}^{n_i} d_j$ where $b_0^i \leq d_0 \leq c_0^i$ and for all $j > 0$, $k_j b_j^i \leq d_j \leq k_j c_j^i$ where k_j is the number of times the jth loop is taken in the PTA. As a consequence, the set of durations of runs reaching ℓ_f is exactly S_1.

We build $\mathcal{A}_2$ similarly. We now build the PTA $\mathcal{A}$ which can either immediately (with $x = 0$) go to the initial state of $\mathcal{A}_1$ or go immediately to a private location ℓ_{priv} before immediately reaching the initial state of $\mathcal{A}_2$. The final location of $\mathcal{A}_1$ and $\mathcal{A}_2$ are then fused in a single location ℓ_f. We thus have that, the set of runs reaching ℓ_{priv} on the way to ℓ_f are exactly the ones reaching ℓ_f in $\mathcal{A}_2$ (with a prefix of duration 0). And similarly, the set of runs avoiding ℓ_{priv} on the way to ℓ_f are exactly the ones reaching ℓ_f in $\mathcal{A}_1$ (with a prefix of duration 0). Therefore, for any parameter valuation v, we have that $\mathit{DVisit}^{priv}(v(\mathcal{A})) = \mathit{DVisit}^{\overline{priv}}(v(\mathcal{A}))$ iff $S_1(v) = S_2(v)$, concluding the reduction.

◀

B.8 Proof of Theorem 31

▶ **Theorem 31.** *The* $\exists$*OE* *problem is decidable.*

Proof. Within the proof of Theorem 28, we considered two expressions $\bar{e}_{\ell_{priv}}$ and $\bar{e}_{\neg\ell_{priv}}$ such that (Proposition 26) $\bar{e}_{\ell_{priv}} = PET(\mathcal{A}_{\ell_\mathrm{f}}^{\ell_{priv}})$ and $\bar{e}_{\neg\ell_{priv}} = PET(\mathcal{A}_{\ell_\mathrm{f}}^{\neg\ell_{priv}})$. Those two expressions were simplified into terms of the form

$$\bar{+}_i (\mathbf{C}_i^d \cap \mathbf{C}_i^{\mathbb{P}}) \bar{\cdot} (\mathbf{C}_1^i)^{\bar{*}} \bar{\cdot} (\mathbf{C}_2^i)^{\bar{*}} \bar{\cdot} \cdots \bar{\cdot} (\mathbf{C}_{m_i}^i)^{\bar{*}}.$$

where every inequality in $\mathbf{C}_j^i$ involves d.

Assume $\bar{e}_{\ell_{priv}}$ is of the above form, and that for all i, j with $j \leq m_i$, $\mathbf{C}_i^{\mathbb{P}} = \bigwedge_k I_{i,-1,k}$, $\mathbf{C}_i^x = \bigwedge_k I_{i,0,k}$, $\mathbf{C}_j^i = \bigwedge_k I_{i,j,k}$ where each $I_{i,r,k}$ is a linear inequality over $\mathbb{P}$ and d.

We build the formula with free variables $d, p_1, \dots, p_M$,

$$\begin{aligned}
\phi_{\ell_{priv}} = \bigvee_i \exists x_0, \dots x_{m_i}, d = \sum_{k=1}^{m_i} x_i \\
\wedge \bigwedge_k I_{i,-1,k}(p_1, \dots, p_M) \\
\wedge \bigwedge_k I_{i,0,k}(x_0, p_1, \dots, p_M) \\
\wedge \bigwedge_j \exists y_1, y_2, y_3, z_1, z_2 (\bigwedge_{m \in \{1,2,3\}} \bigwedge_k I_{i,j,k}(y_m, p_1, \dots, p_M)) \\
\wedge (z_1 = 0 \vee y_1 \mid z_1) \wedge (z_2 = 0 \vee y_2 \mid z_2) \wedge x_j = z_1 + z_2 + y_3.
\end{aligned}$$

For fixed values of the variables $p_1, \dots, p_M$, the set of variables x satisfying $\phi_{\ell_{priv}}$ is exactly the set of integers contained in $\bar{e}_{\ell_{priv}}$ for parameter valuations $p_1, \dots, p_m$.

Indeed, let us fix one value of i; by definition, the conjunction of constraint $\bigwedge_k I_{i,-1,k}(p_1, \dots, p_M)$ constrains the variables $p_1, \dots, p_M$ as $\mathbf{C}_i^{\mathbb{P}}$ does to the parameter valuations. Moreover, by definition of $\bar{\cdot}$, the concatenation of the other constraints accepts the values that can be obtained as a sum of elements produced by each constraint. This is the role played by the variables x_i in the formulas.

The main point to show is that for $j \geq 1$, the variable x_j takes exactly the values accepted by $(\mathbf{C}_j^i)^{\bar{*}}$. Remember that $(\mathbf{C}_j^i)^{\bar{*}}$ accepts every number obtained as a sum of terms accepted by $\mathbf{C}_j^i$.

First, by definition, y_1, y_2 and y_3 all satisfy $\mathbf{C}_j^i$. Thus, z_1 and z_2, being integer multiple of y_1 and y_2, satisfy $(\mathbf{C}_j^i)^{\bar{*}}$. Hence, any possible value of x_j belongs to $(\mathbf{C}_j^i)^{\bar{*}}$.

Reciprocally, let $n \in \mathbb{N}$ accepted by $(\mathbf{C}_j^i)^{\bar{*}}$. There thus exist $n_1, \dots, n_k$ such that for all r, n_r satisfies $\mathbf{C}_j^i$ and $n = \sum_{r=1}^k n_r$. Assume $n_1 \leq n_2 \leq \dots \leq n_r$. By convexity of the set described by $\mathbf{C}_j^i$, every integer between n_1 and n_r satisfies the constraint. Thus, we can assume w.l.o.g. that at most one number n_s has a value strictly between n_1 and n_r (if two such numbers a and b exist, one can replace them by $a+1$ and $b-1$ to bring them closer to n_1 and n_r, and by repeating this process, at most one remains). There thus exist $v_1, v_r \in \mathbb{N}$ and $v \in [n_1; n_r]$ such that $n = v_1 n_1 + v_r n_r + v$. By setting $y_1 = n_1$, $y_2 = n_r$, $z_1 = v_1 n_1$, $z_2 = v_r n_r$ and $y_3 = v$, the variable x_j takes the value n.[2]

We build $\phi_{\ell_{pub}}$ from $\bar{e}_{\ell_{pub}}$ in the same way. Asking whether there exist parameter valuations $p_1, \dots, p_M$ such that an integer $d \in \mathbb{N}$ appears in both $\bar{e}_{\ell_{pub}}$ and $\bar{e}_{\ell_{pub}}$ is thus equivalent to verifying the truth of the formula

$$\exists p_1, \dots, p_M, d, \phi_{\ell_{pub}}(d, p_1, \dots, p_M) \wedge \phi_{\ell_{priv}}(d, p_1, \dots, p_M).$$

As this formula belongs to the existential fragment of Presburger arithmetic with divisibility, its veracity is decidable, and thus $\exists$`OE` is decidable.

◀

B.9 Proof of Theorem 33

▶ **Theorem 33.** *$\exists$`OE` is decidable in* EXPSPACE *for $(1, *, 1)$-PTAs over discrete time.*

[2] The formula allows for $z_1 = 0$ and $z_2 = 0$, so that if n satisfies $\mathbf{C}_j^i$, we can set $z_1 = z_2 = 0$ and $y_3 = n$.

Proof. In [7, Section 8], we gave a semi-algorithm to answer the $\exists$OS problem in $(1, *, 1)$-PTAs, working as follows. We build the parallel composition of two occurrences of the input PTA and, adding an absolute time clock, we force simultaneous reachability of the final location such that one PTA visited ℓ_{priv} while the other did not. This can be reused here, by replacing the absolute time clock with a synchronized action between both PTAs (knowing the actual execution time is not necessary here, as we aim at solving $\exists$OE—not $\exists$OS). Assuming $\mathcal{A}$ is a $(1, *, 1)$-PTA, let $\mathcal{A}'$ denote this resulting PTA. Now, from our construction, $\exists$OE holds iff the final location of $\mathcal{A}'$ is reachable for at least one parameter valuation.

Note that, while the (unique) parametric clock of the PTA must be duplicated in $\mathcal{A}'$, the (unique) parameter is not duplicated, as it is the same in both versions of the PTA, and therefore $\mathcal{A}'$ contains a single parameter. That is, $\mathcal{A}'$ is a $(2, *, 1)$-PTA.

Finally, reachability emptiness is EXPSPACE-complete in $(2, *, 1)$-PTA over discrete time [17], and therefore the $\exists$OE problem for $(1, *, 1)$-PTAs over discrete time can be solved in EXPSPACE. ◀