On the Monniaux Problem in Abstract Interpretation

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IRISA, November 2019

The concept of an invariant is one of the most important in mathematics.

Encyclopedia of Mathematics, Kluwer, 2002

The MU Puzzle

A string-rewriting system using letters **M**, **I**, and **U**

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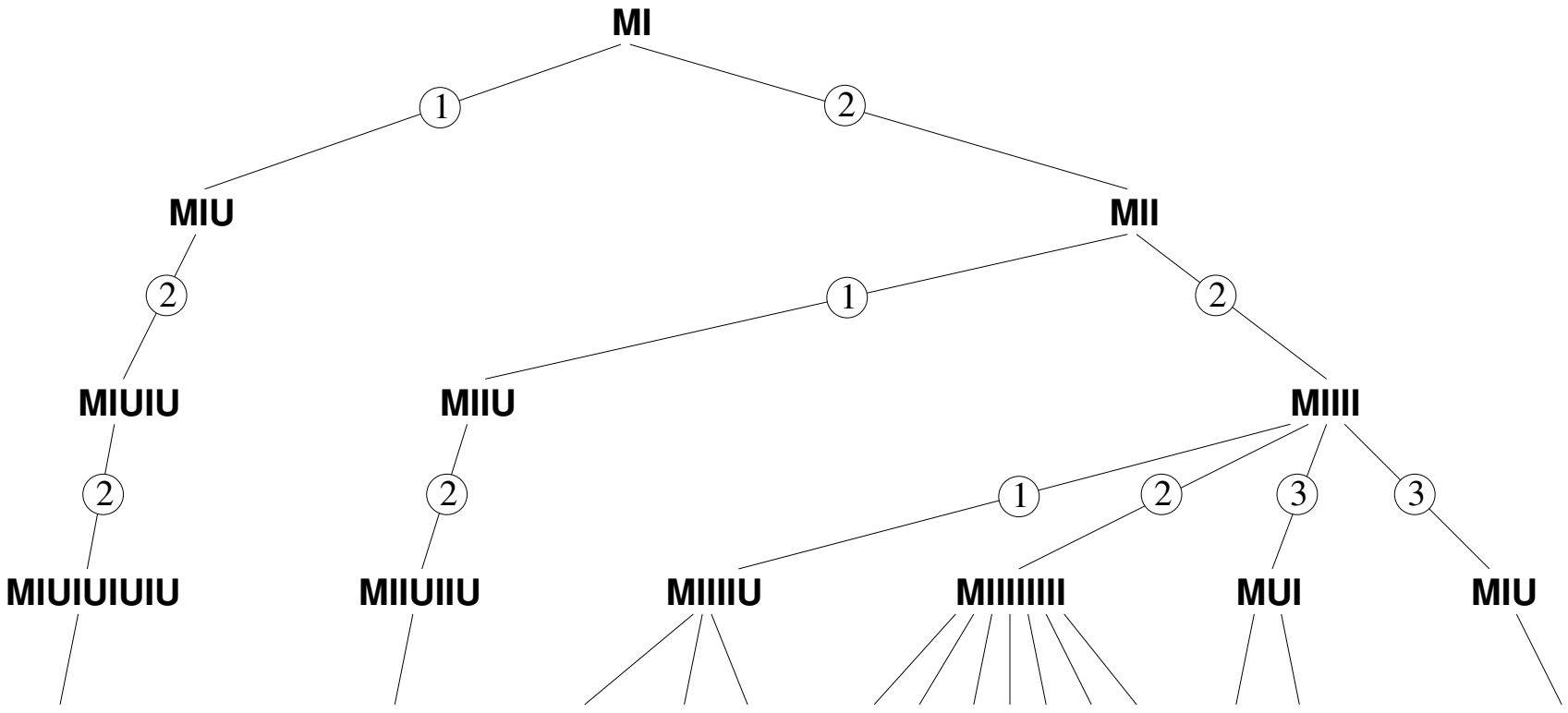
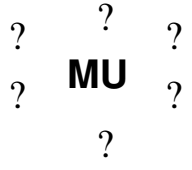
Starting from **MI**, the goal is to produce **MU**

Can It Be Solved?





Can This (Nondeterministic) Program Halt?

$s := \mathbf{MI}$;
while $s \neq \mathbf{MU}$ do
    choose
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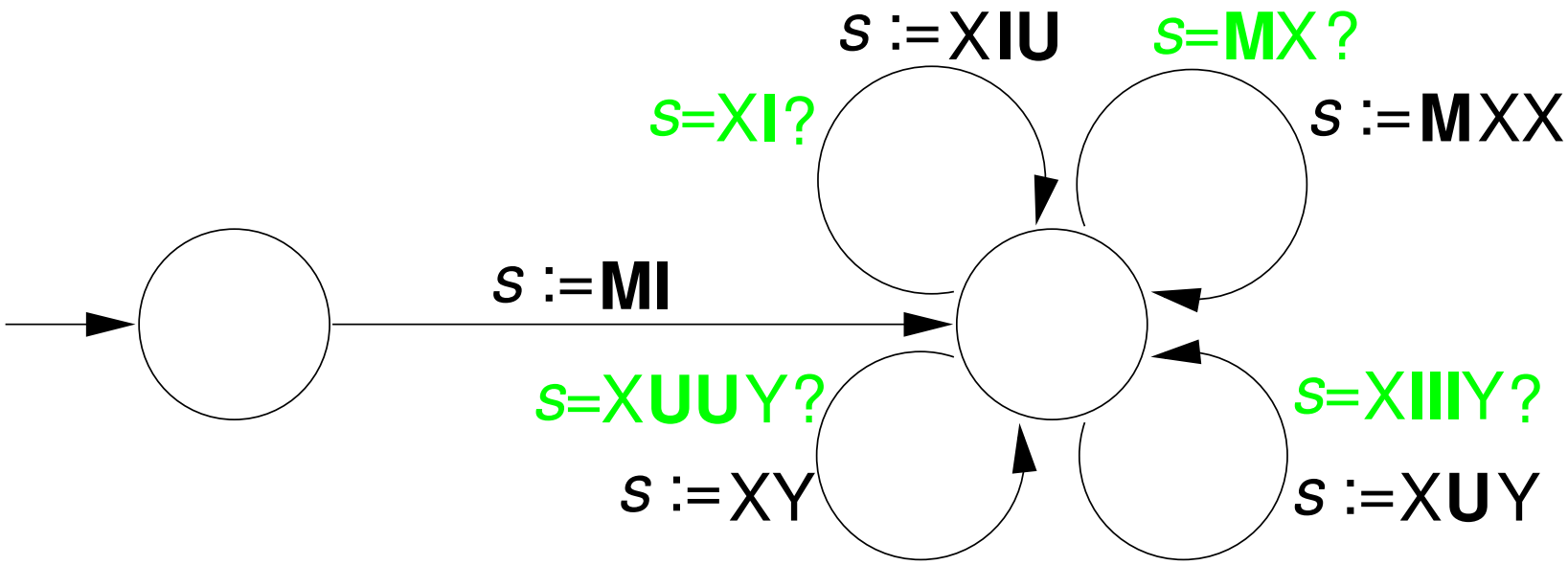

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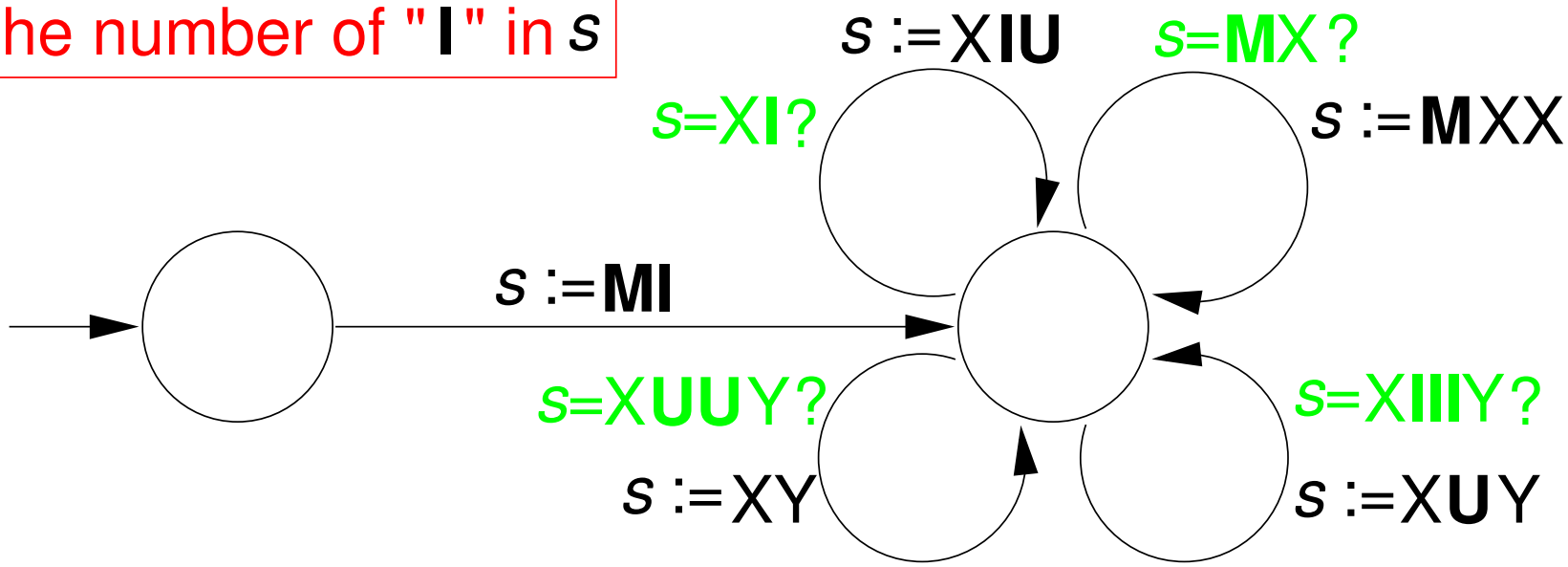
introduce "i" to count the number of "**I**" in s



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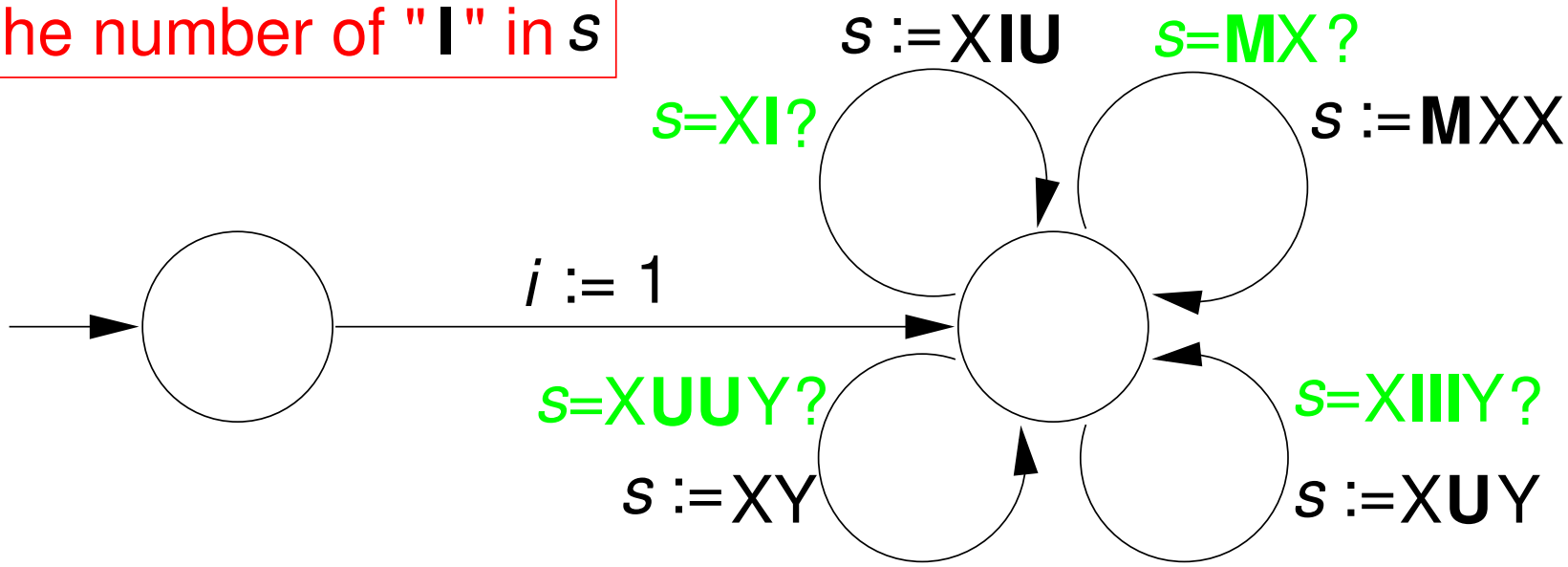

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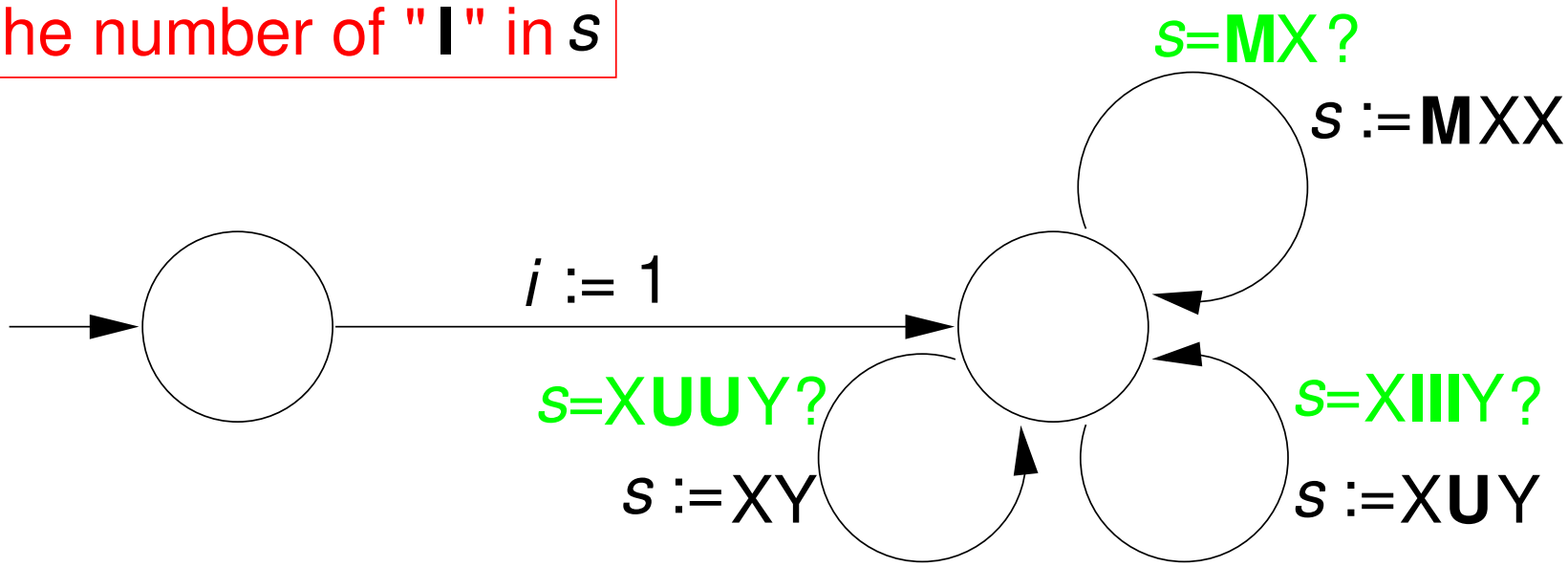

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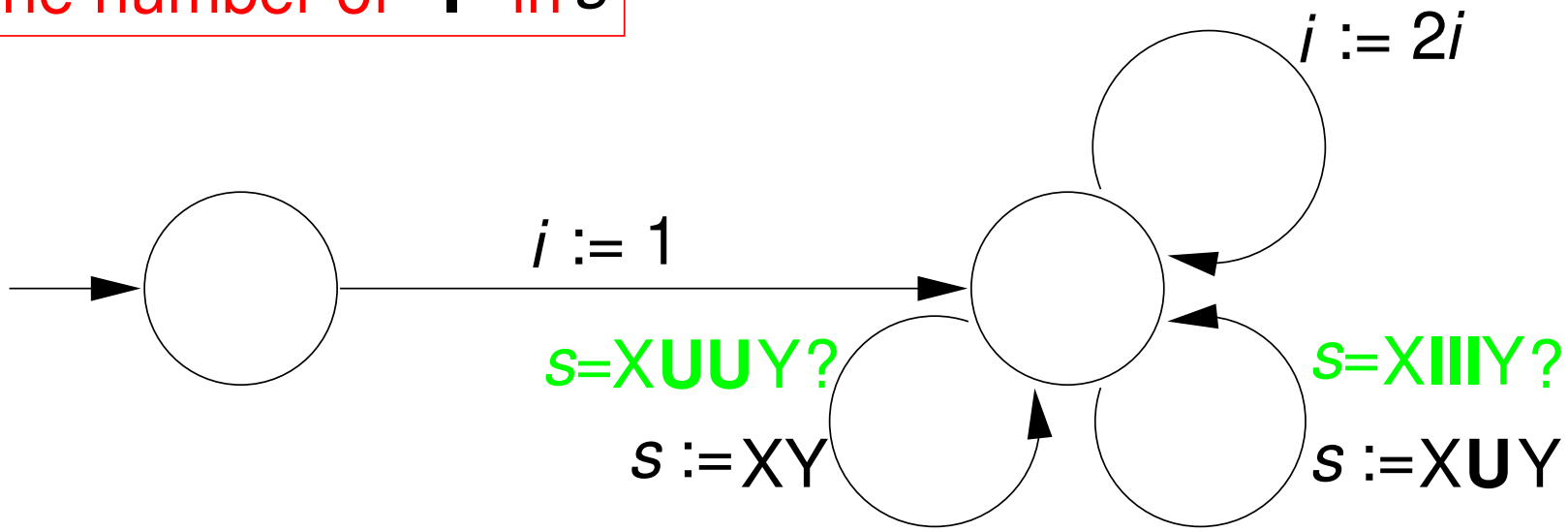

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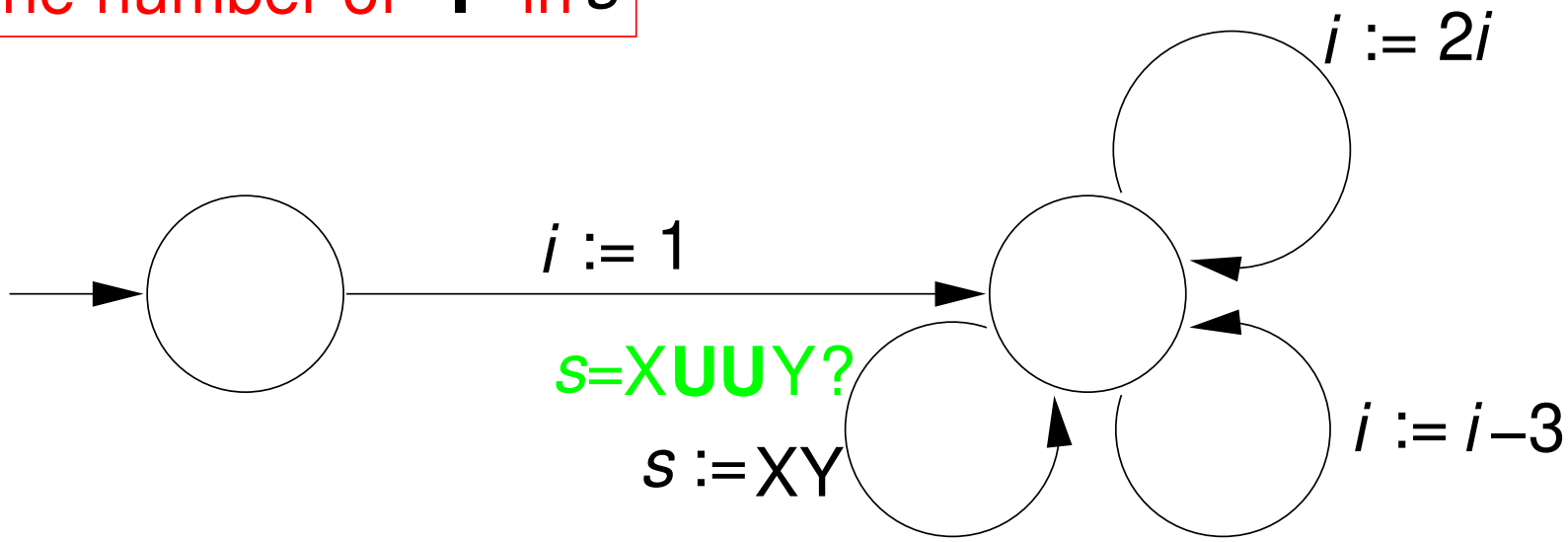

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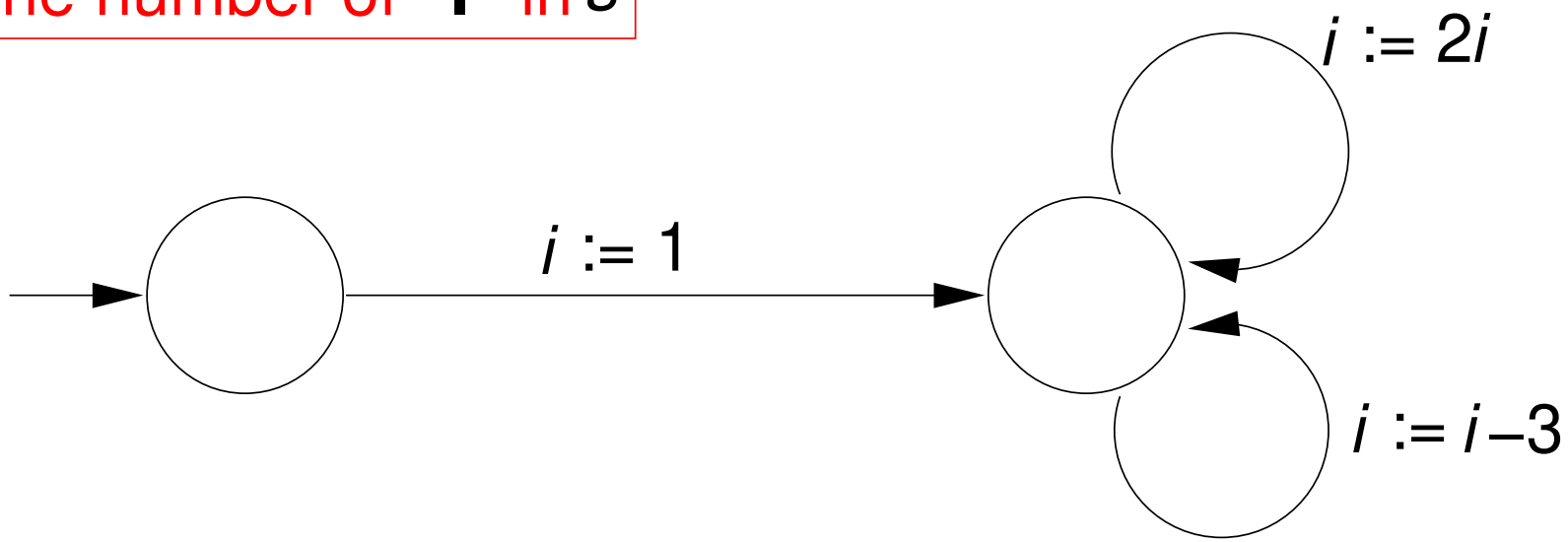

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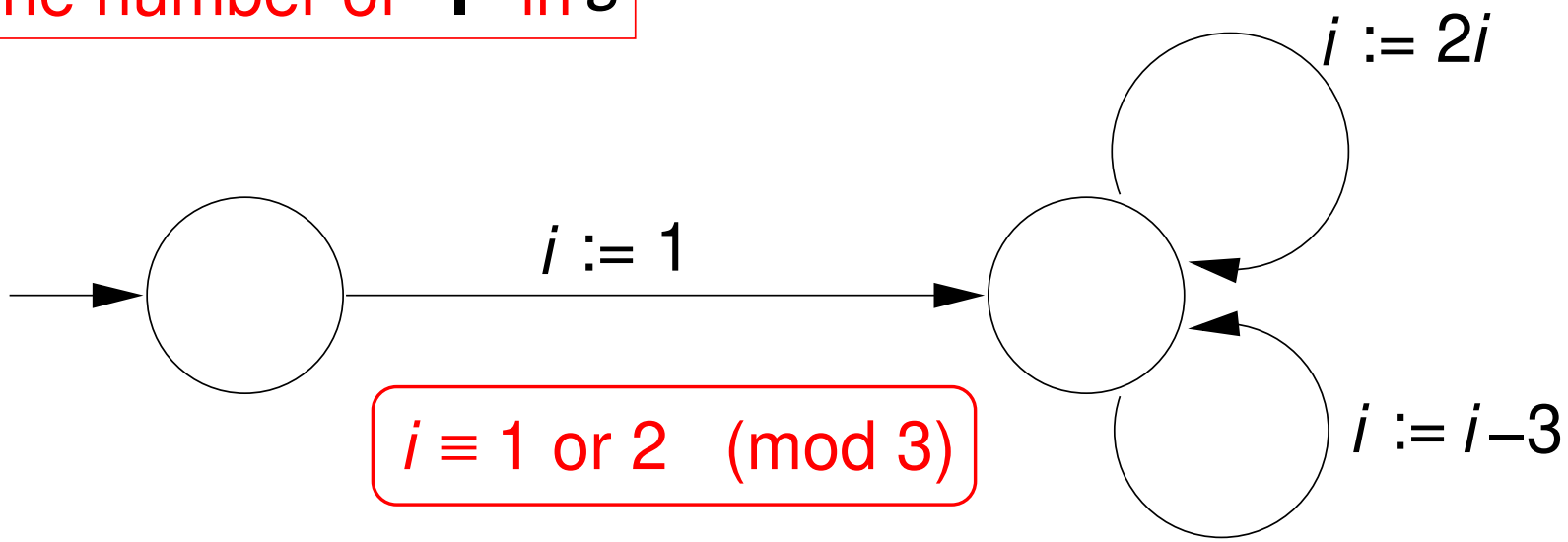
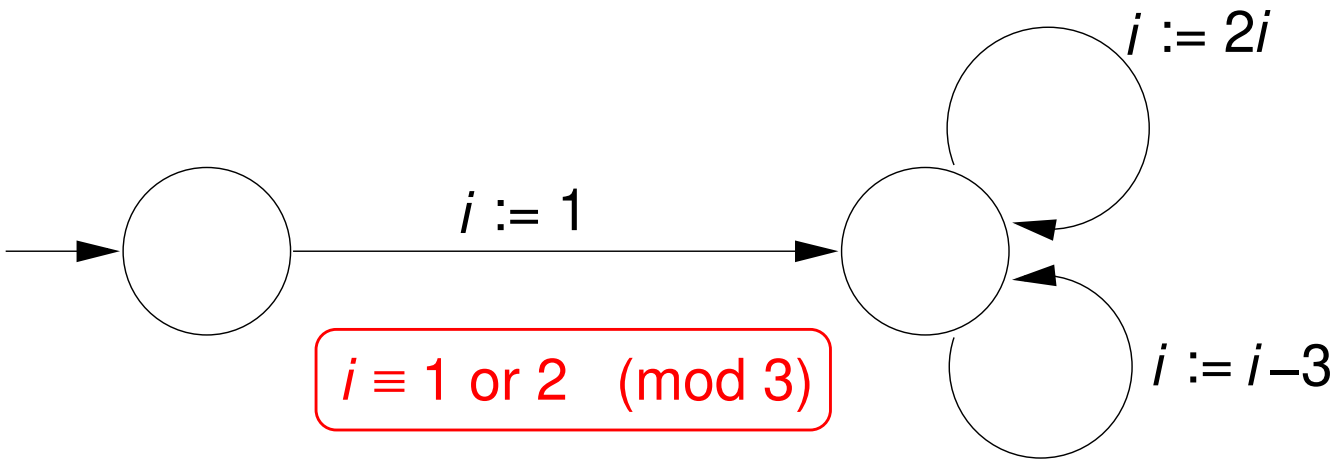


A Polynomial Program Invariant



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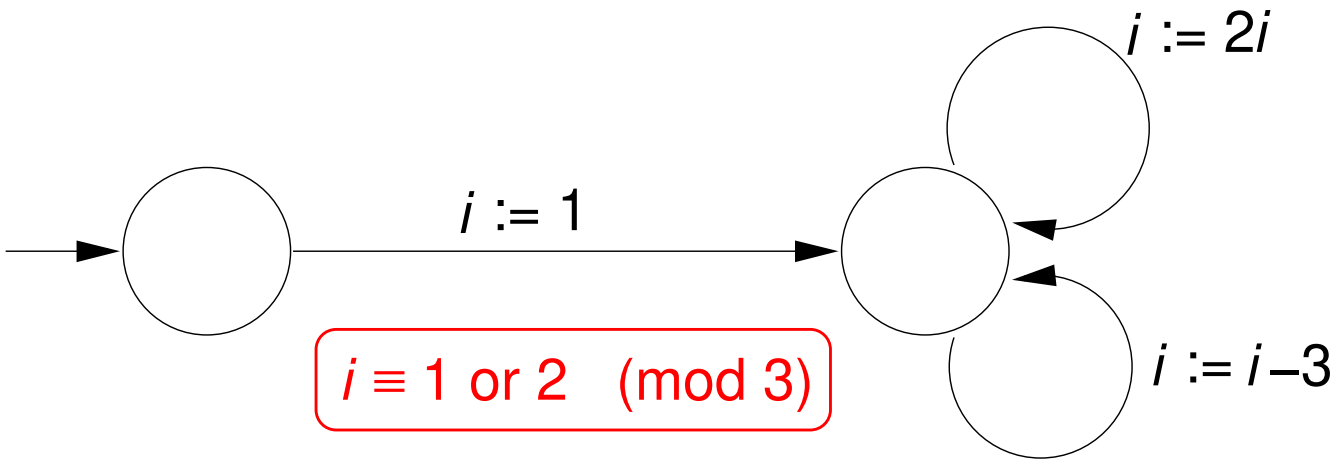

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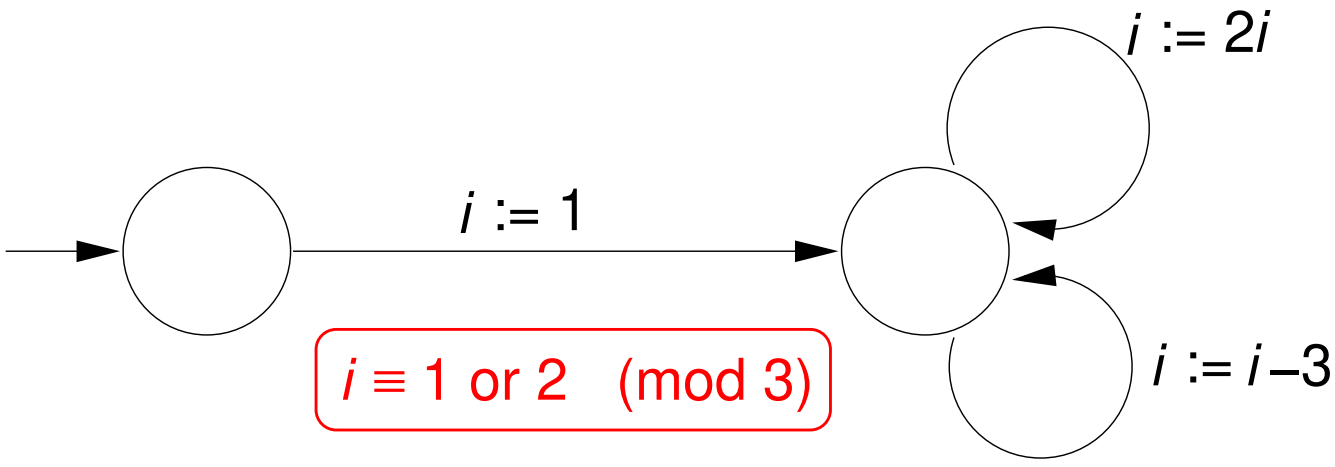

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- Then this program satisfies

$$p(i) = 0$$

(over $\mathbb{Z}_3$)

Automated Invariant Generation: A Challenge

The classical approach to the verification of temporal safety properties of programs requires the construction of **inductive invariants** *at each program point, that is, assertions that are true on every program execution reaching that point, and moreover, that are closed under the strongest postcondition operator.* **Automation of this construction is the main challenge in program verification.**

D. Beyer, T. Henzinger, R. Majumdar, and A. Rybelchenko
Invariant Synthesis for Combined Theories, 2007

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invariant = overapproximation (of the reachable states)

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$$\textbf{inductive invariant} = \begin{cases} \text{overapproximation} \\ \text{preserved by the transition relation} \end{cases}$$

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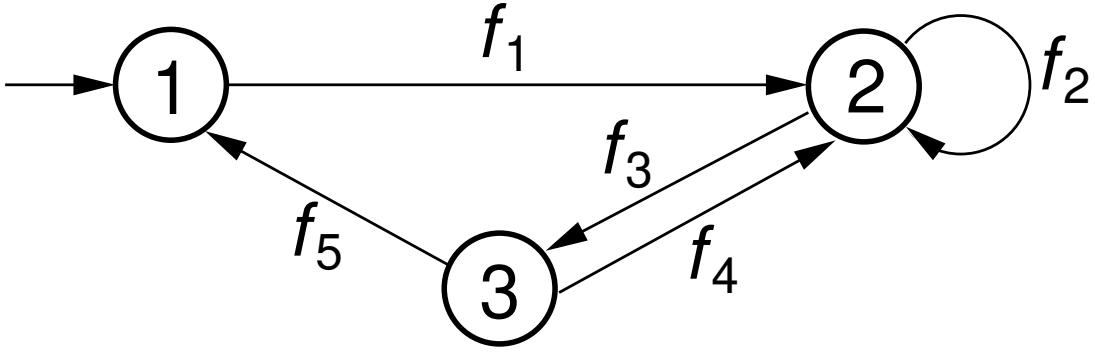

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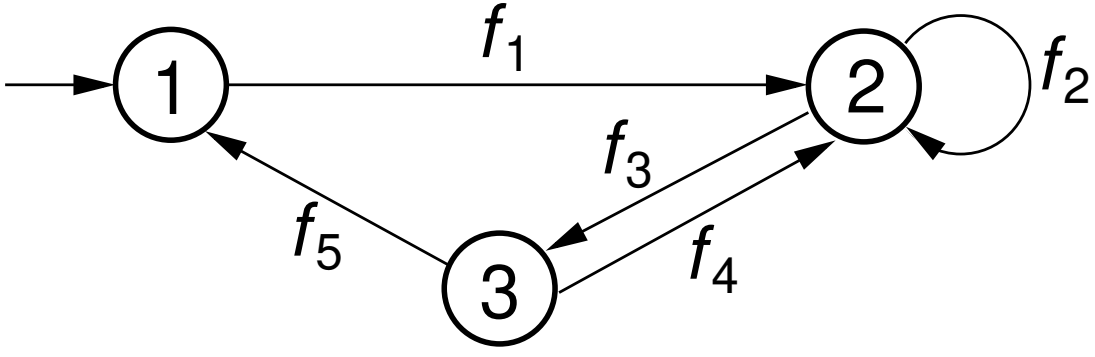
x, y, z range over $\mathbb{Z}$ (or $\mathbb{Q}$)



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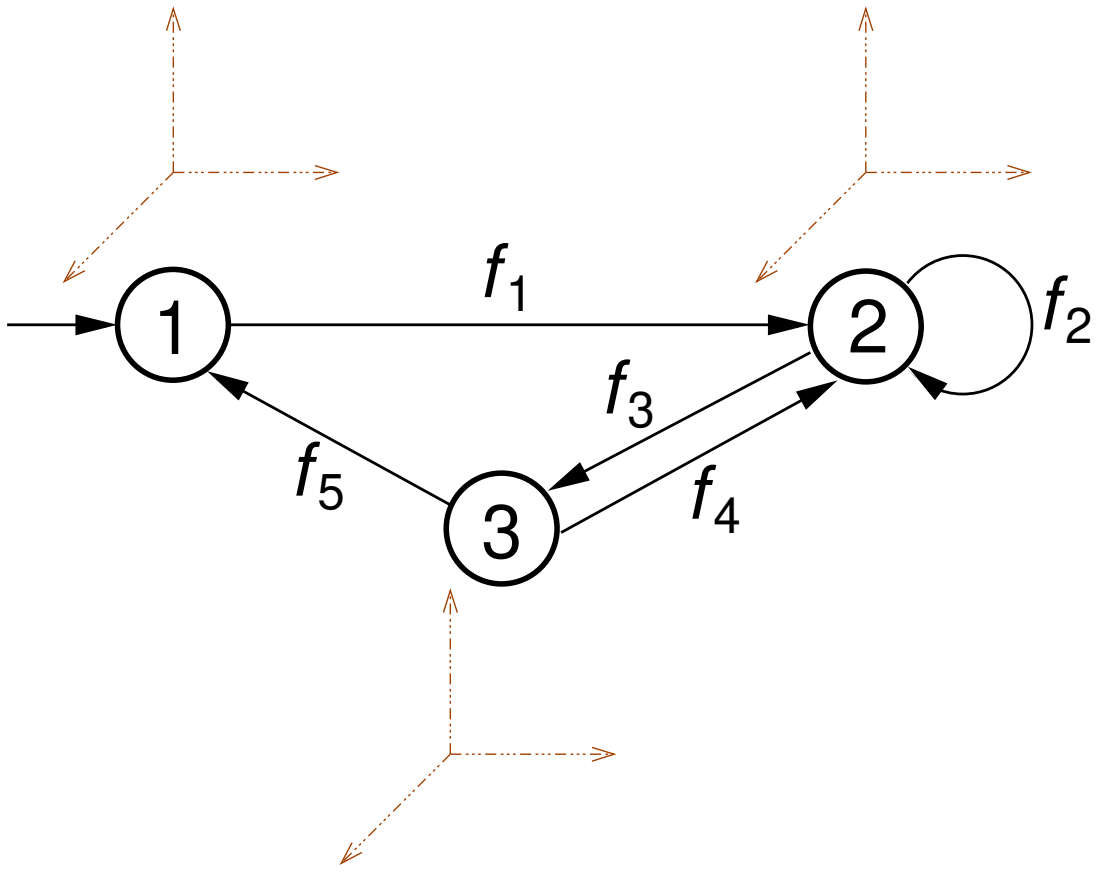

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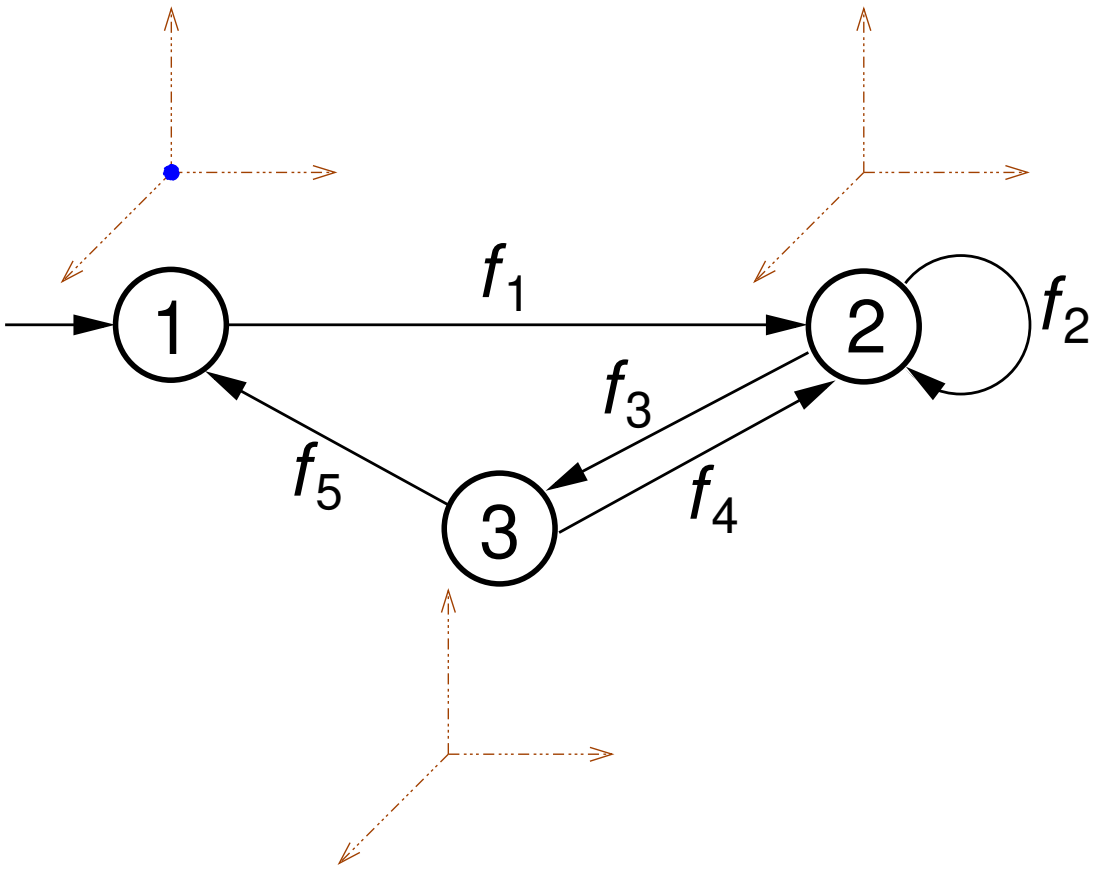

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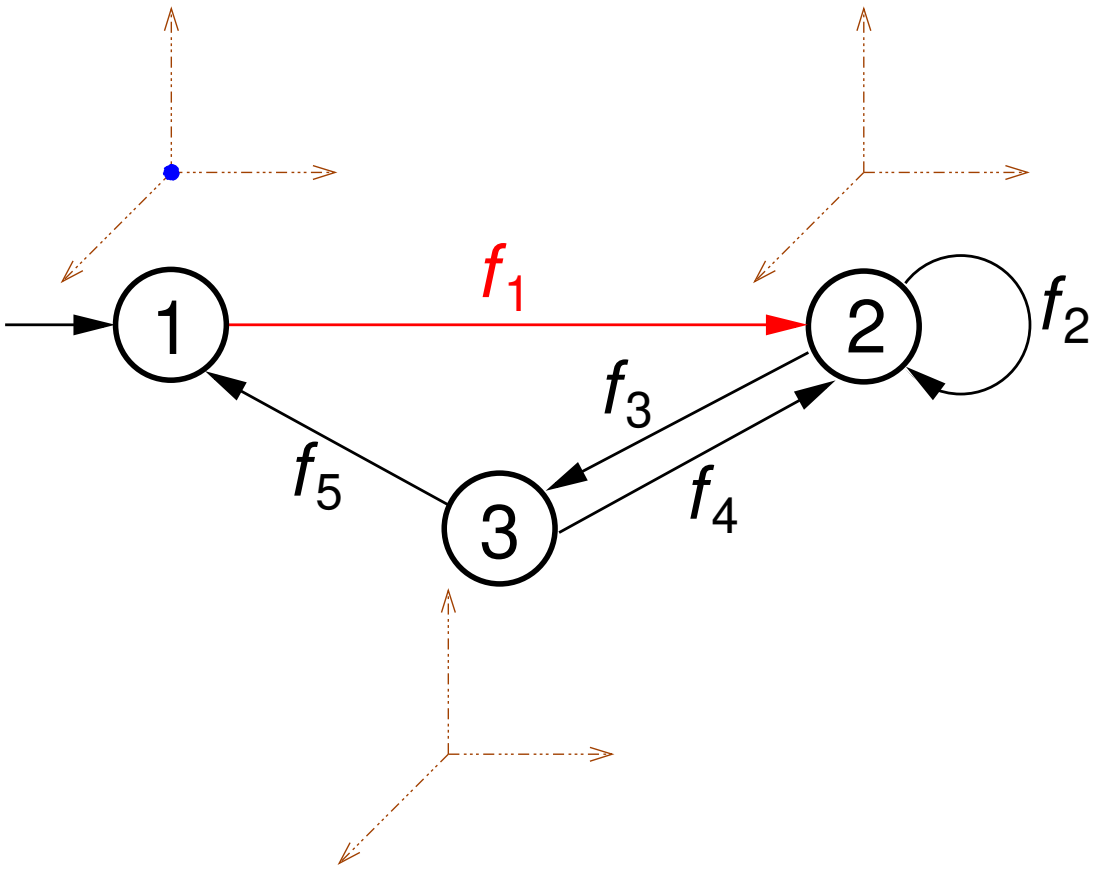

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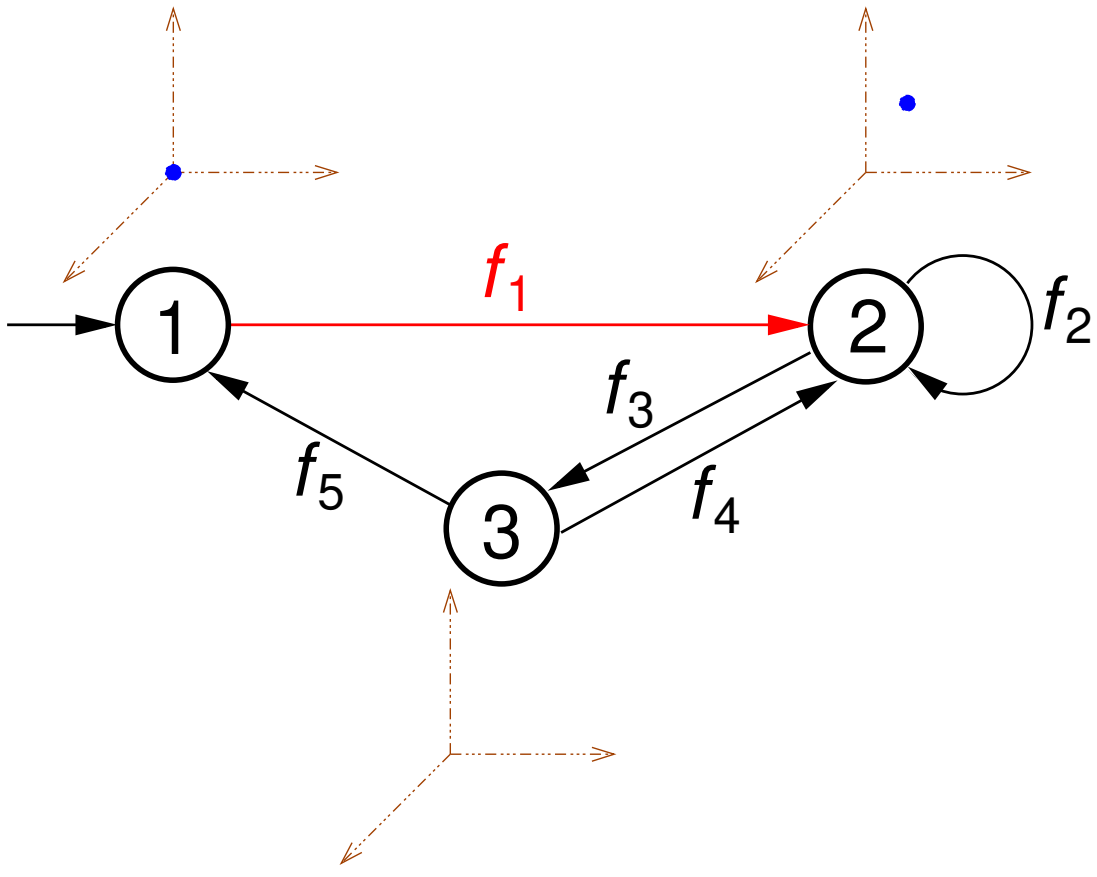

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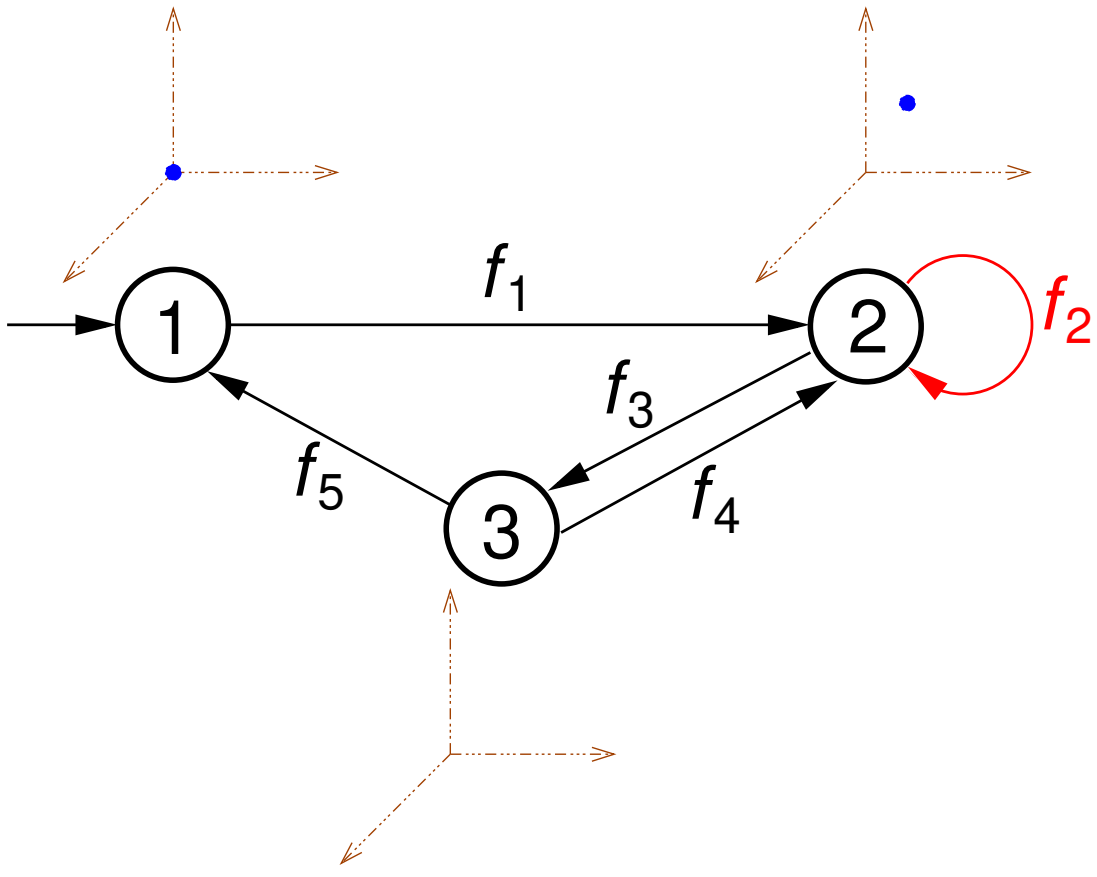

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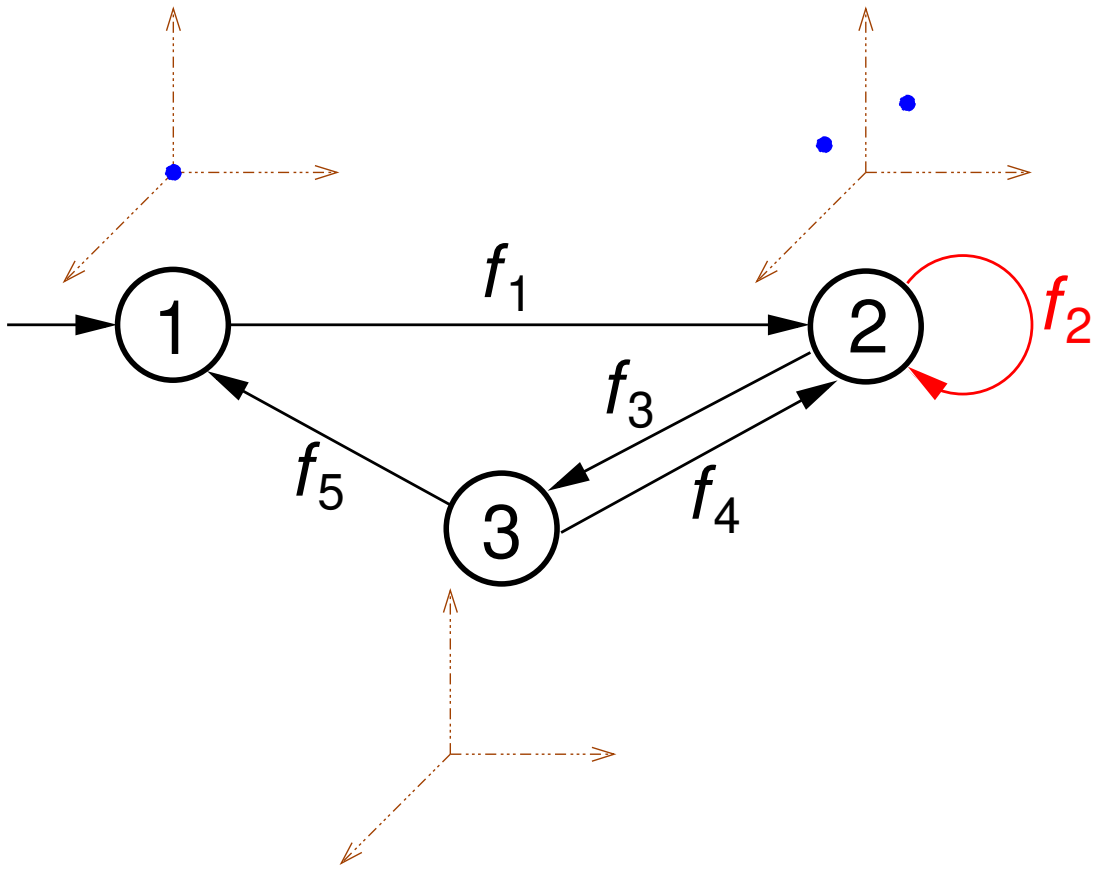

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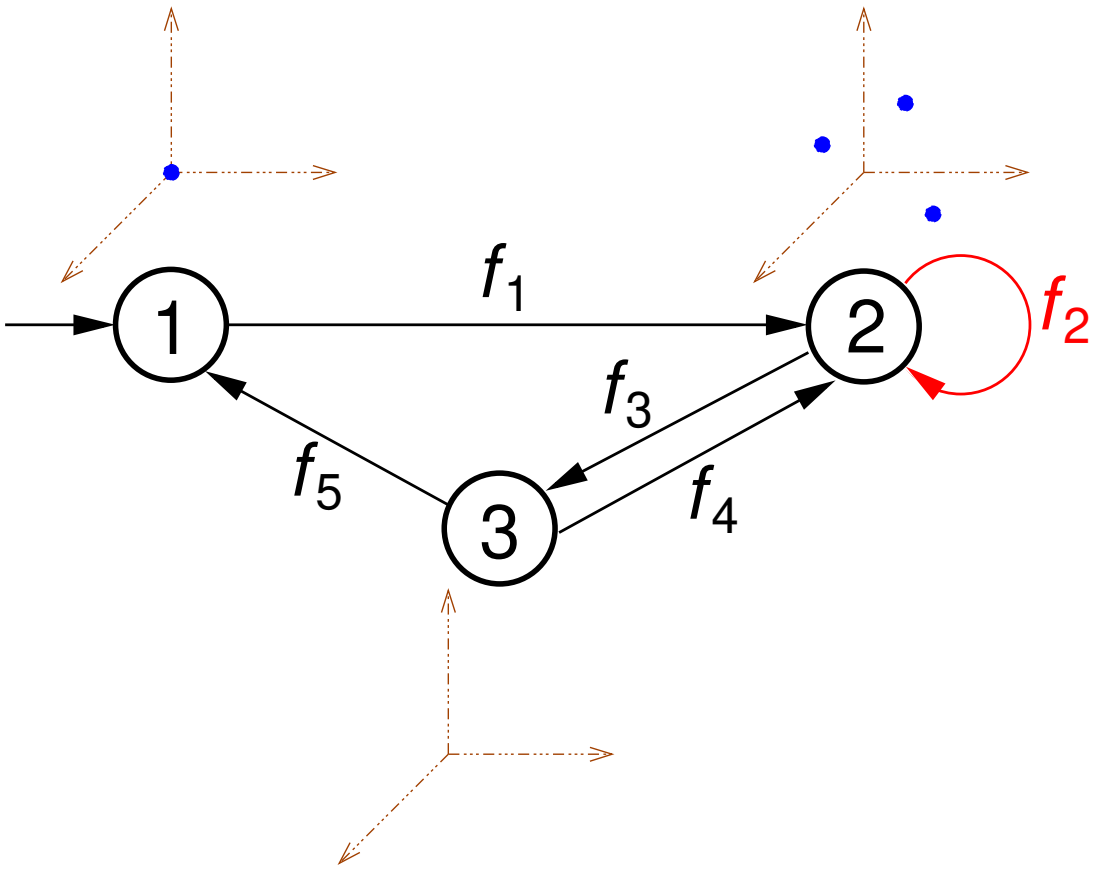

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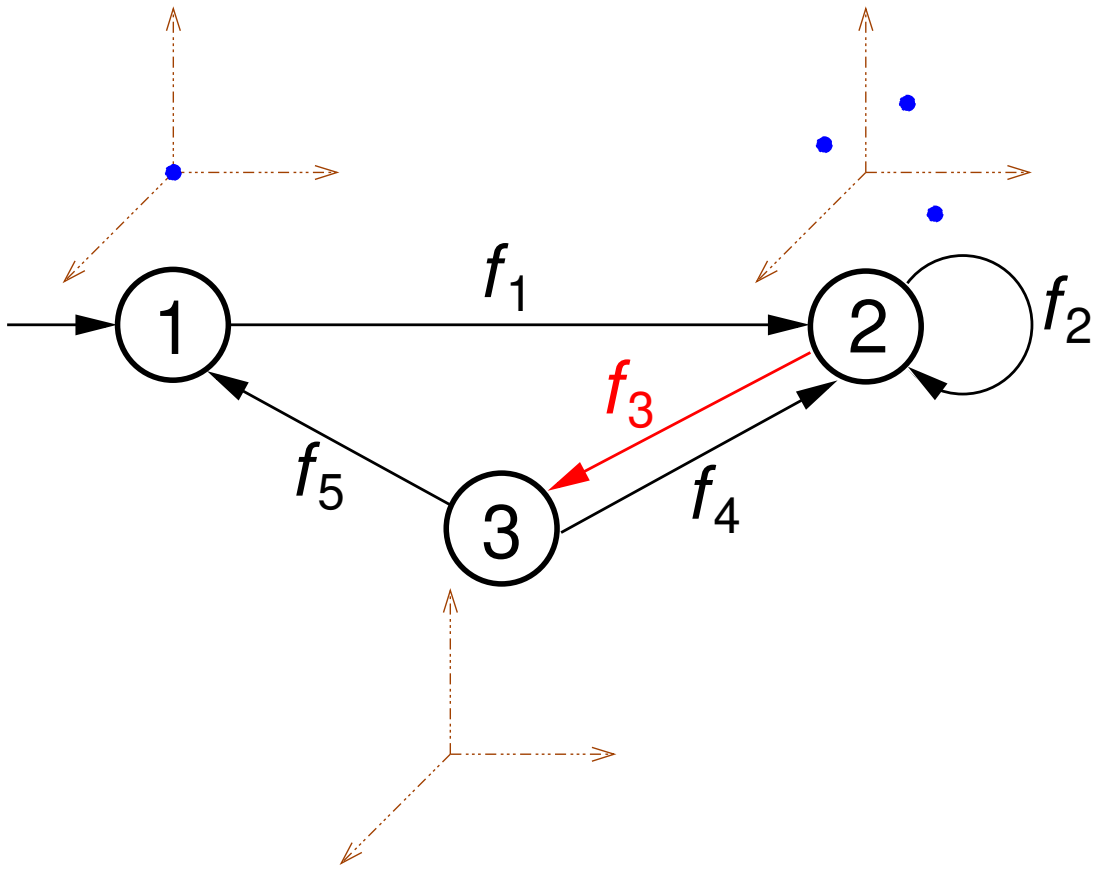

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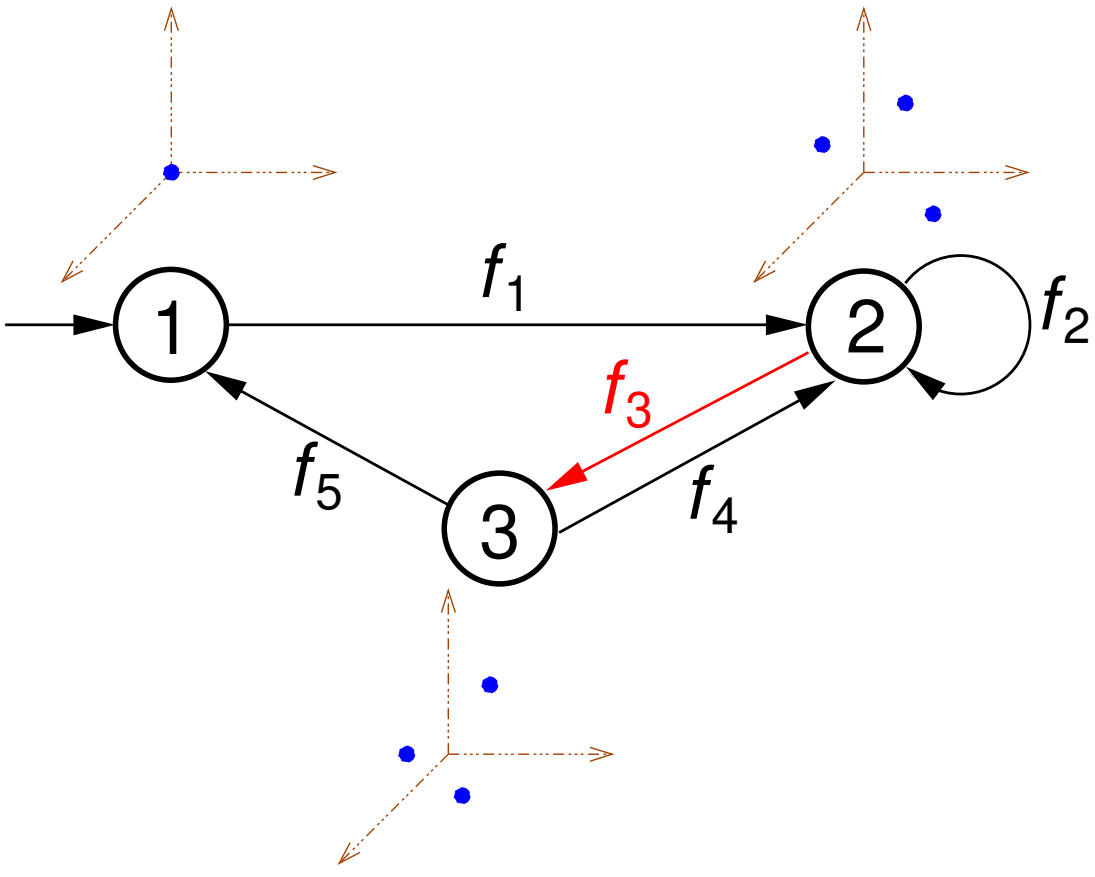

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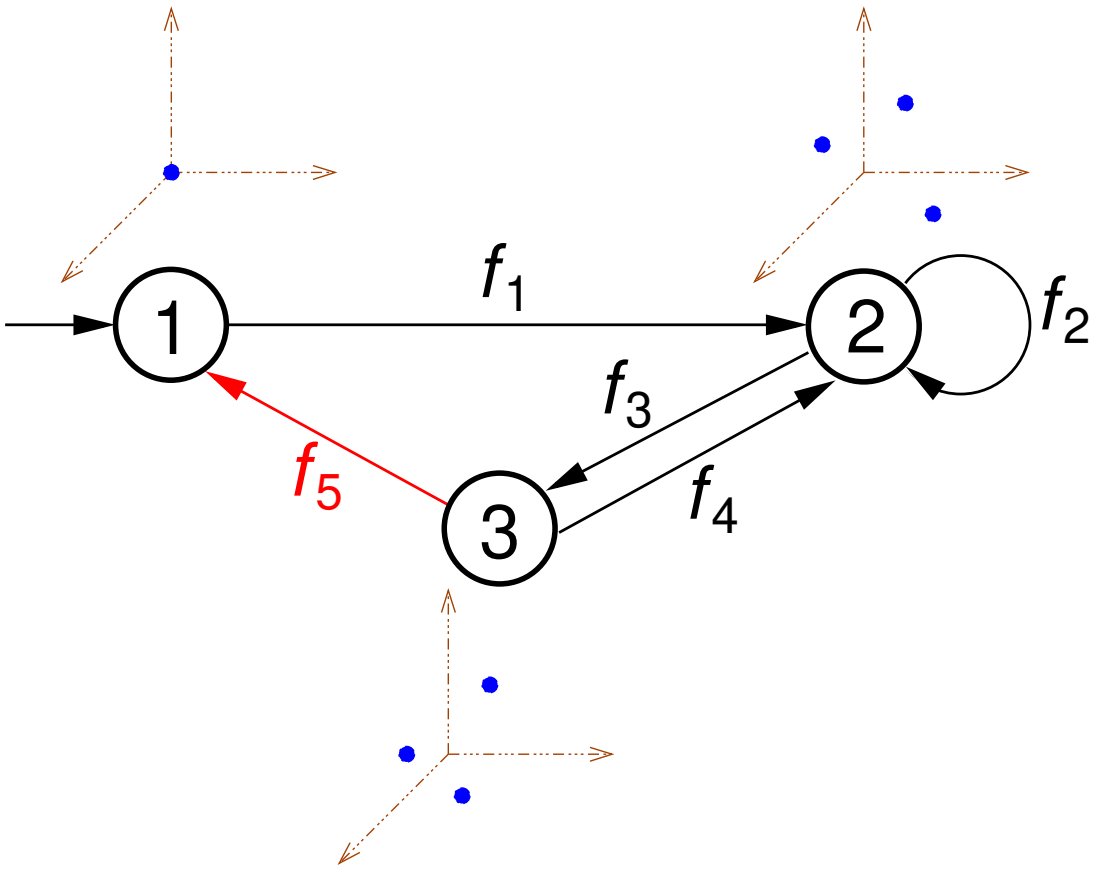

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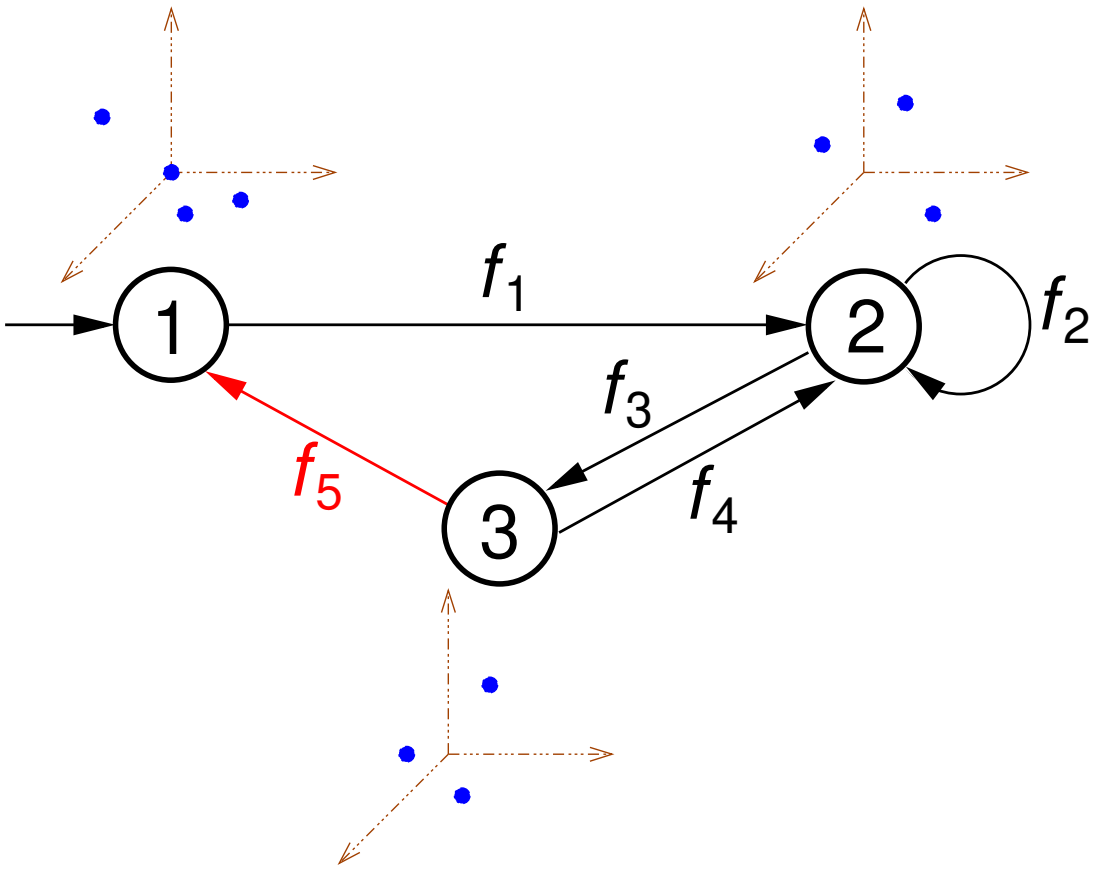

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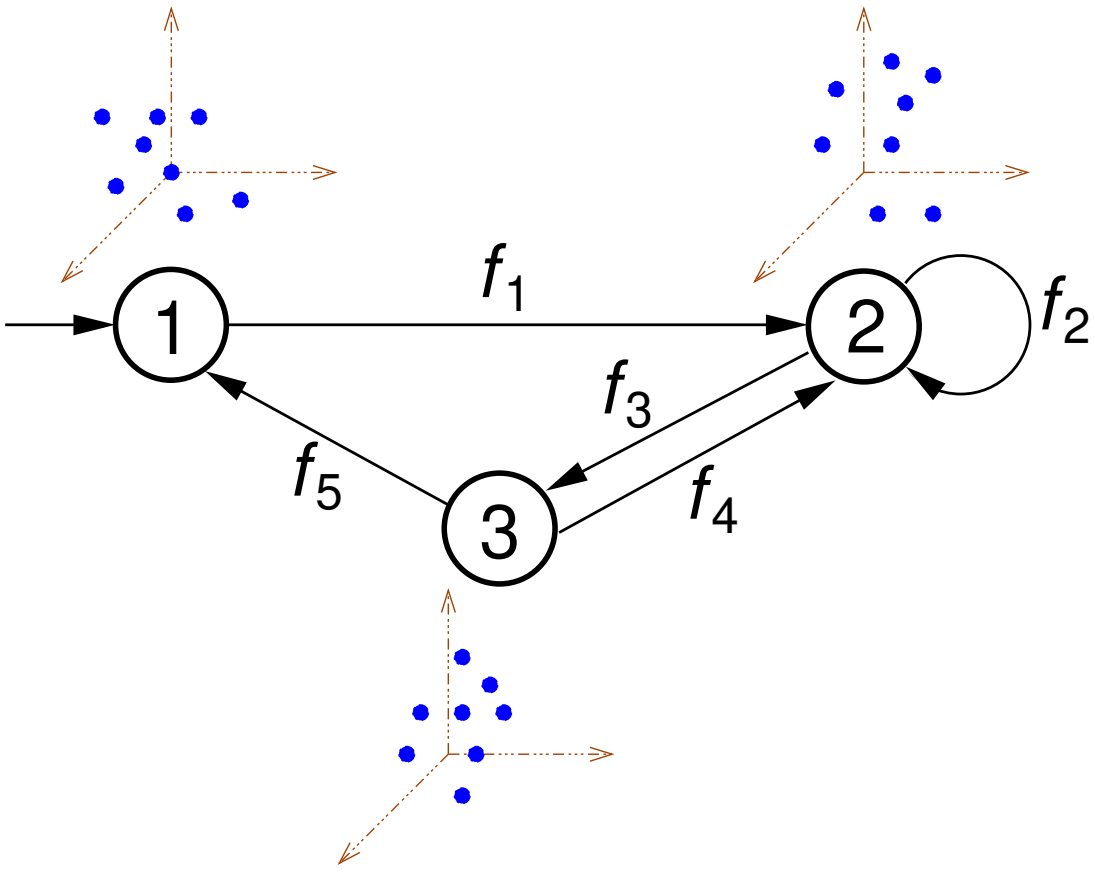

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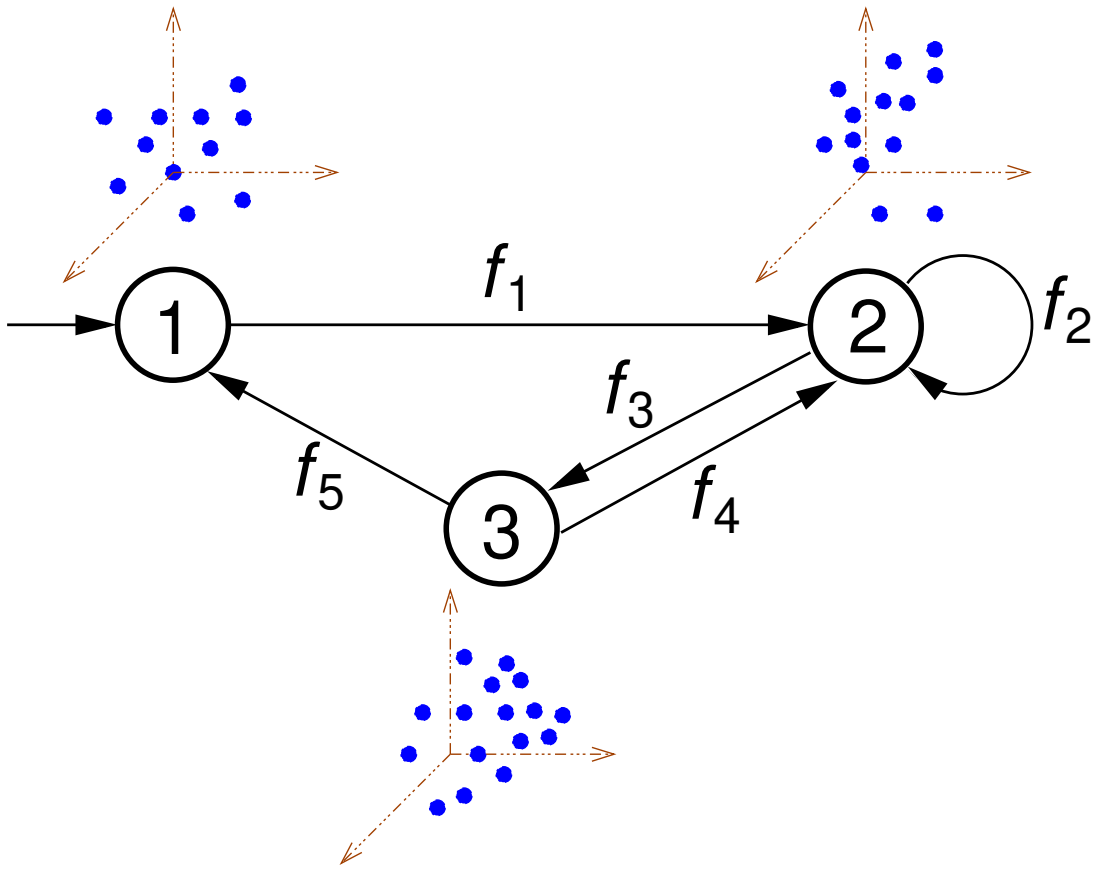

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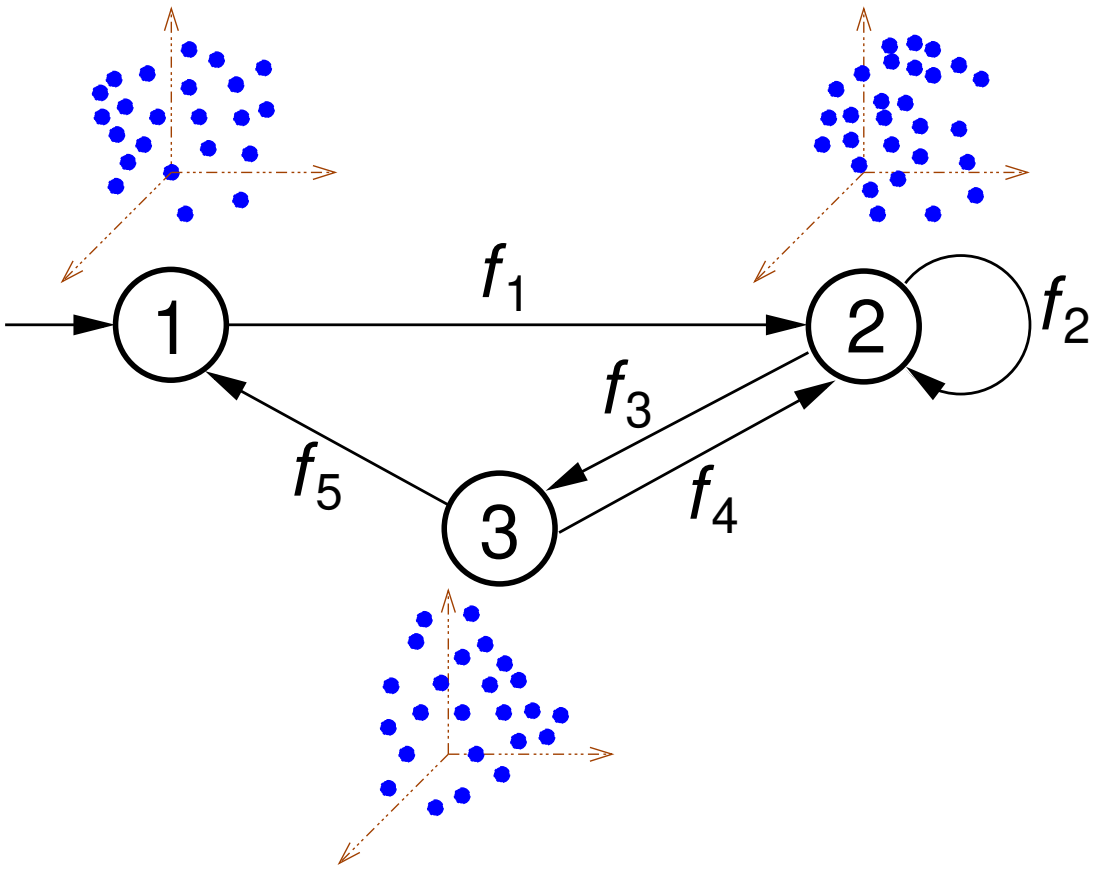

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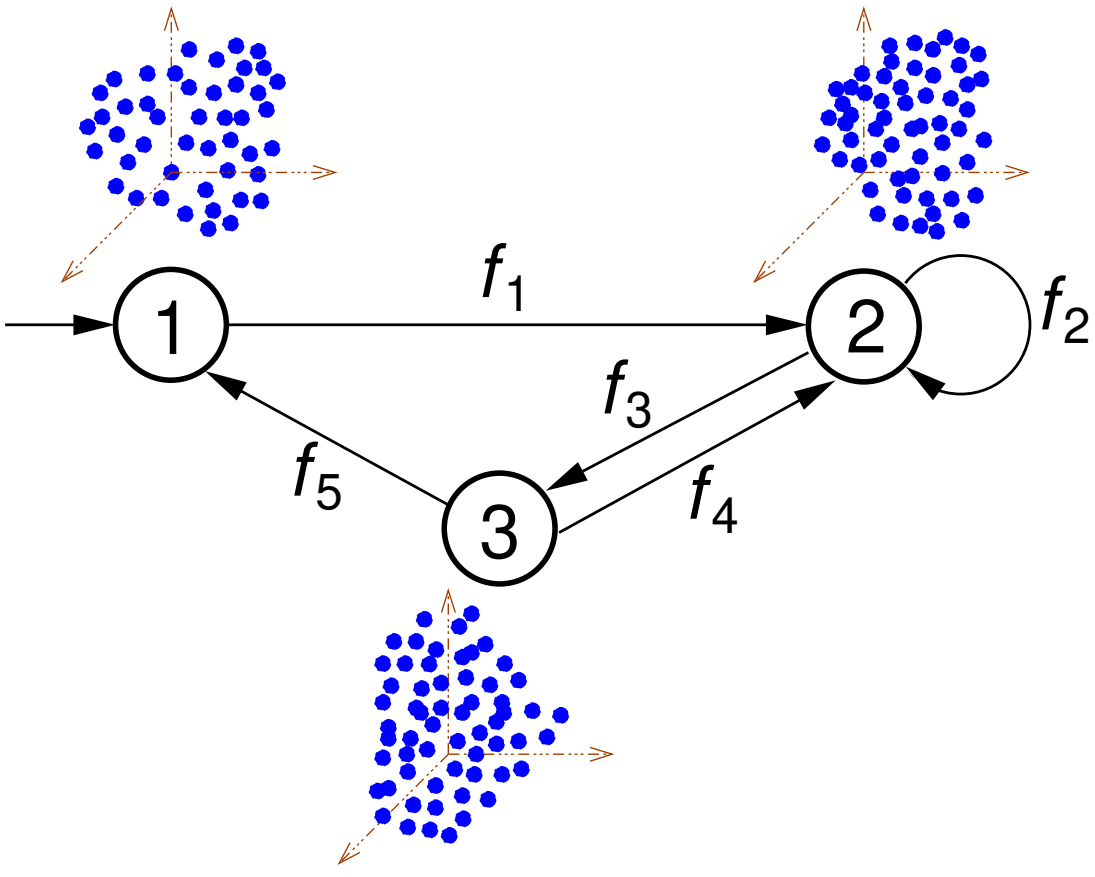

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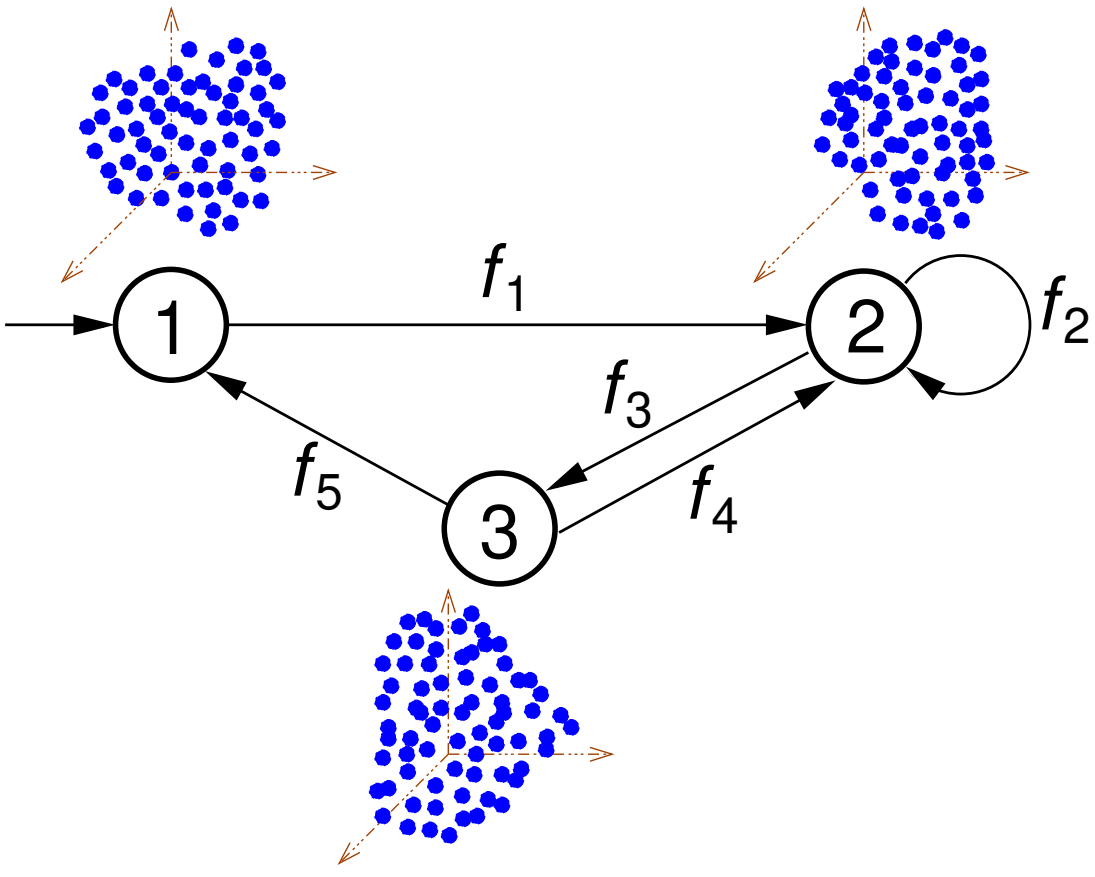

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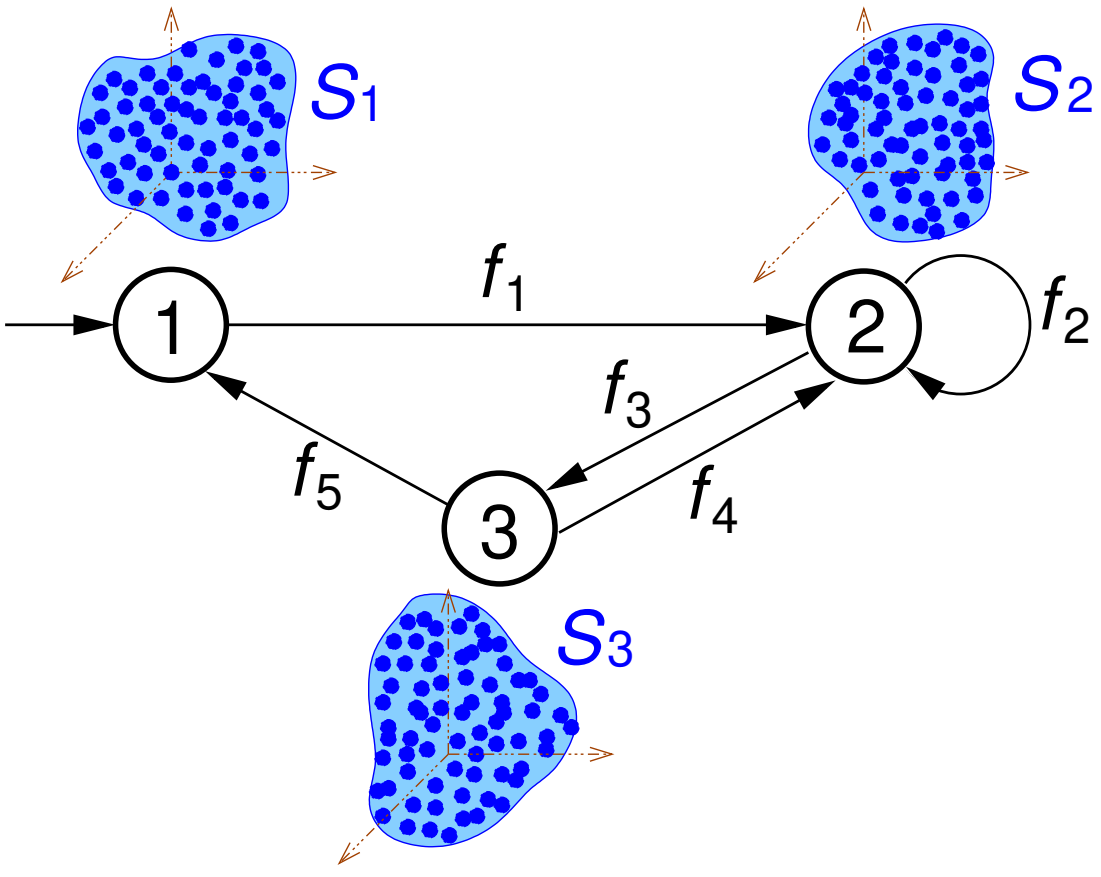

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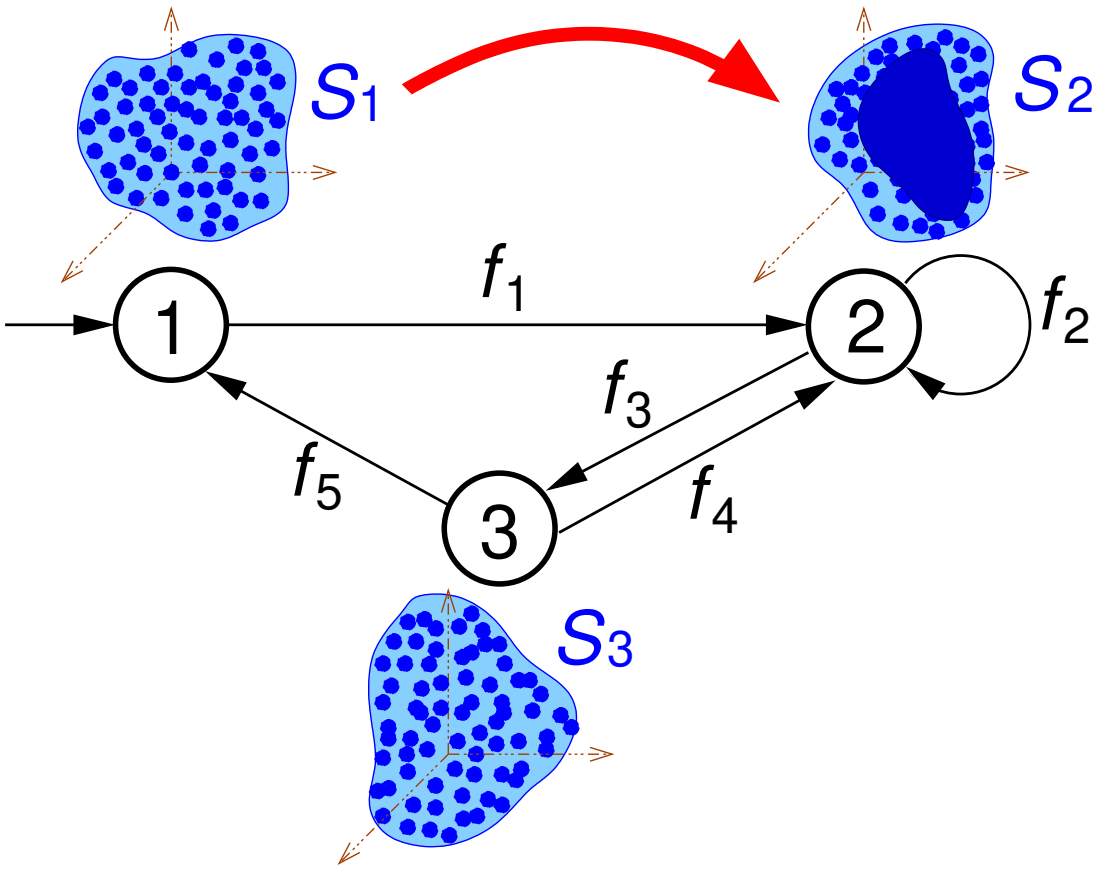

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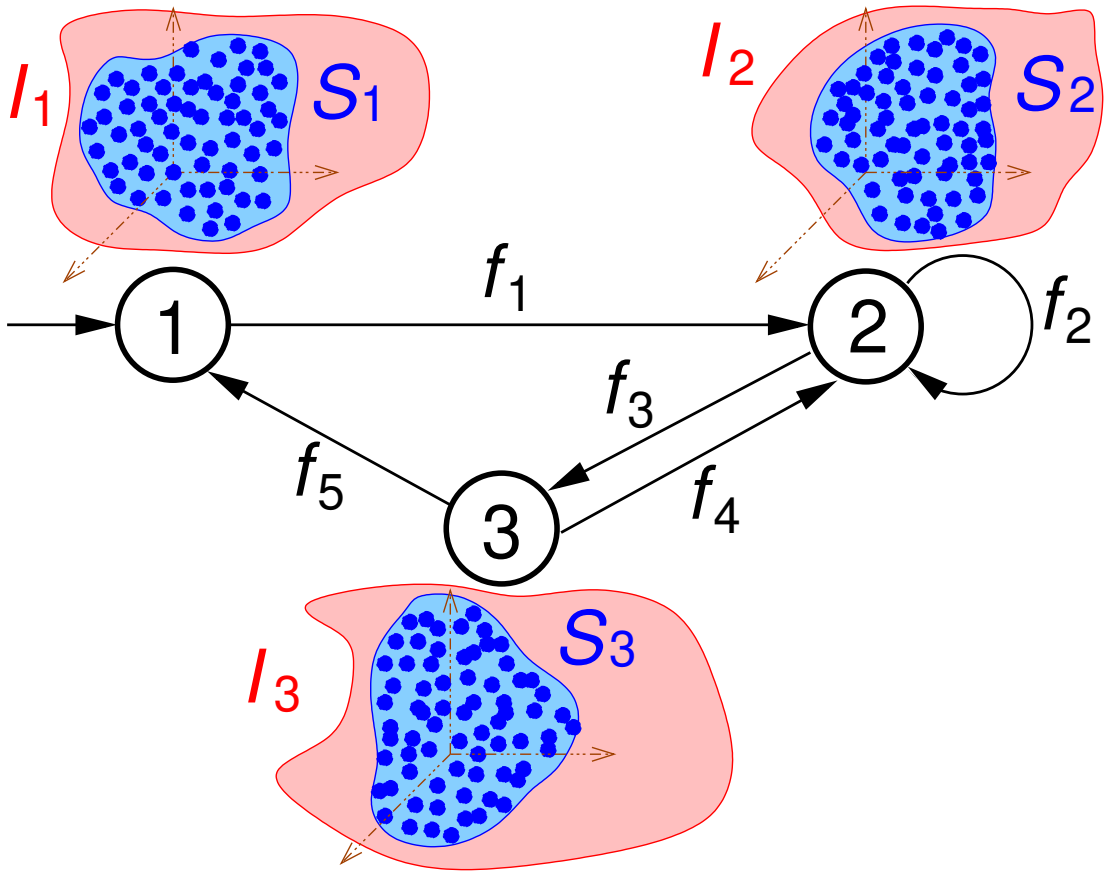

$\langle I_1, I_2, I_3 \rangle$ is an **invariant** ($I_1, I_2, I_3 \subseteq \mathbb{R}^3$)

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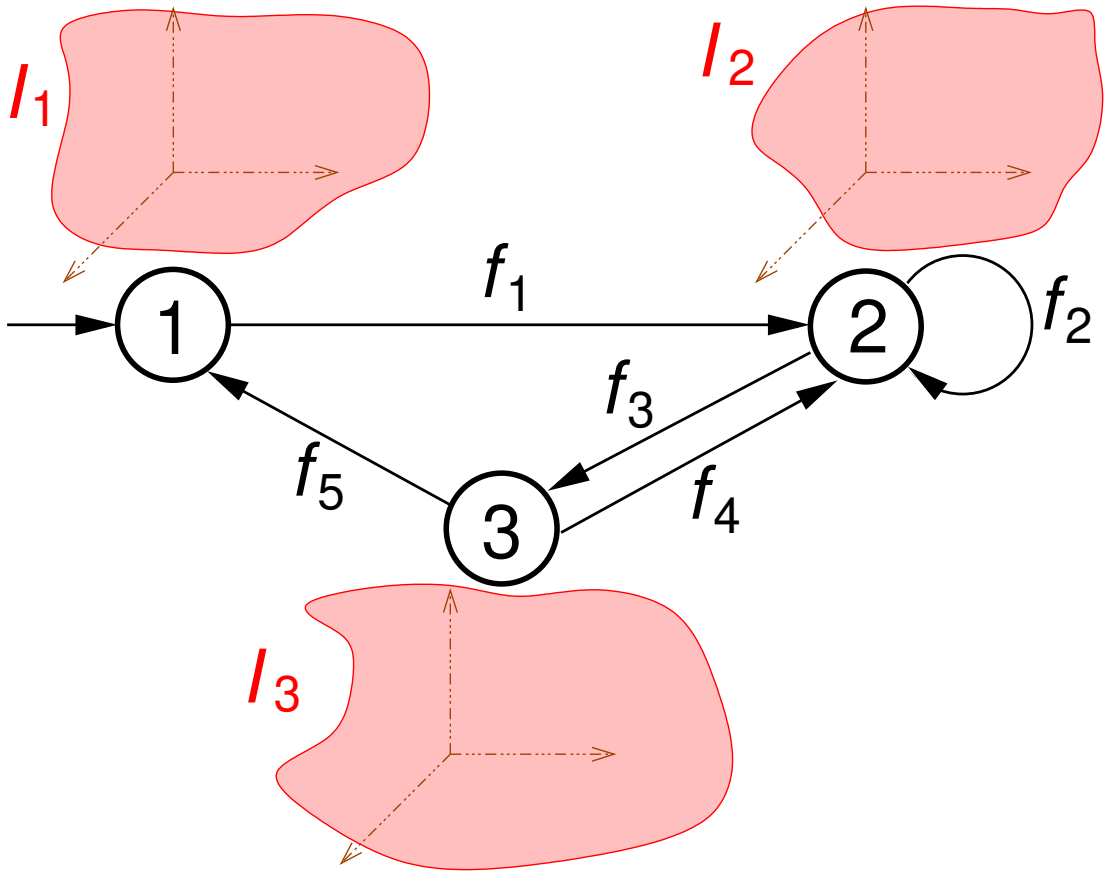

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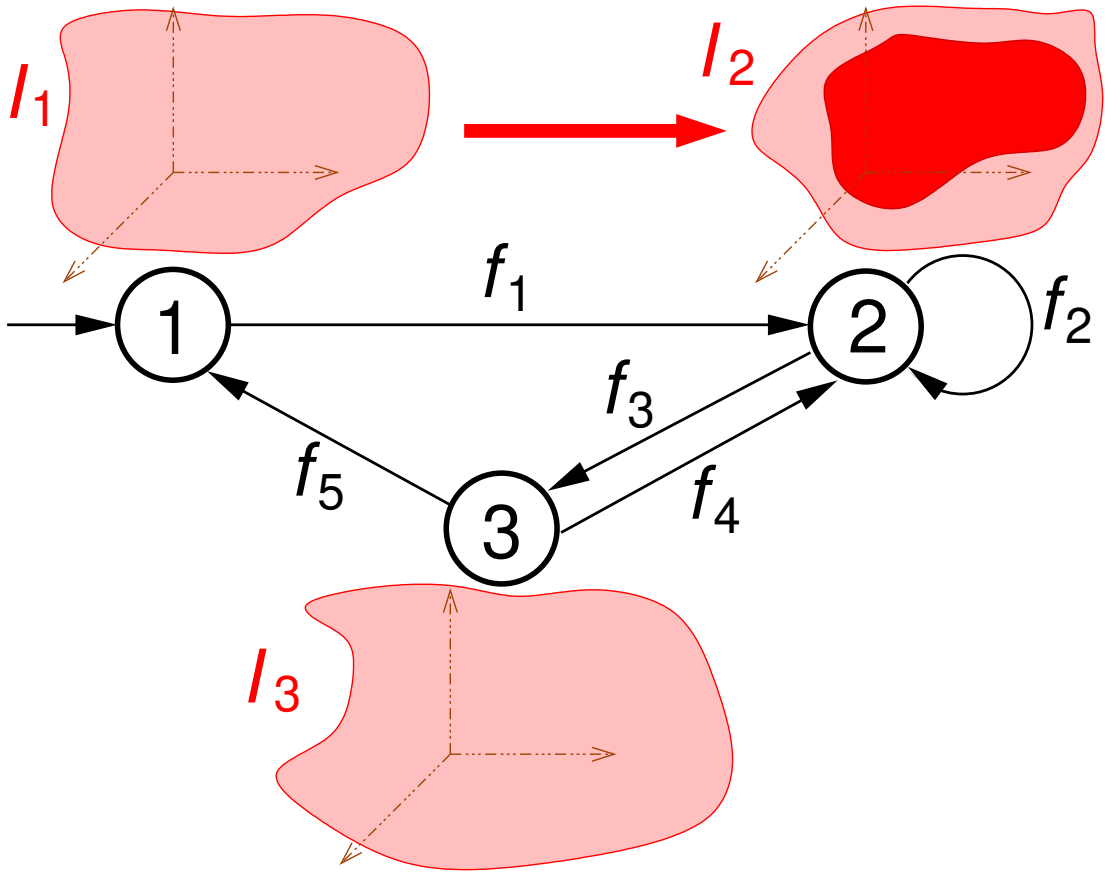

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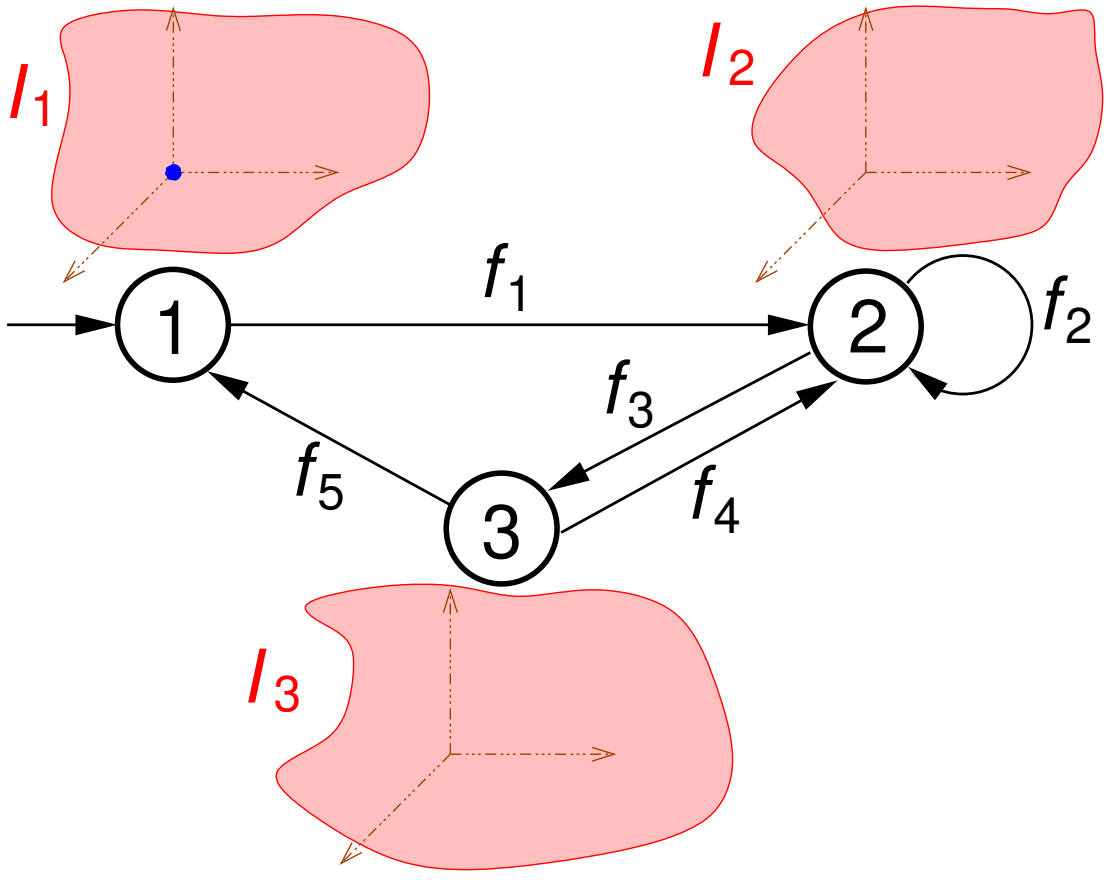

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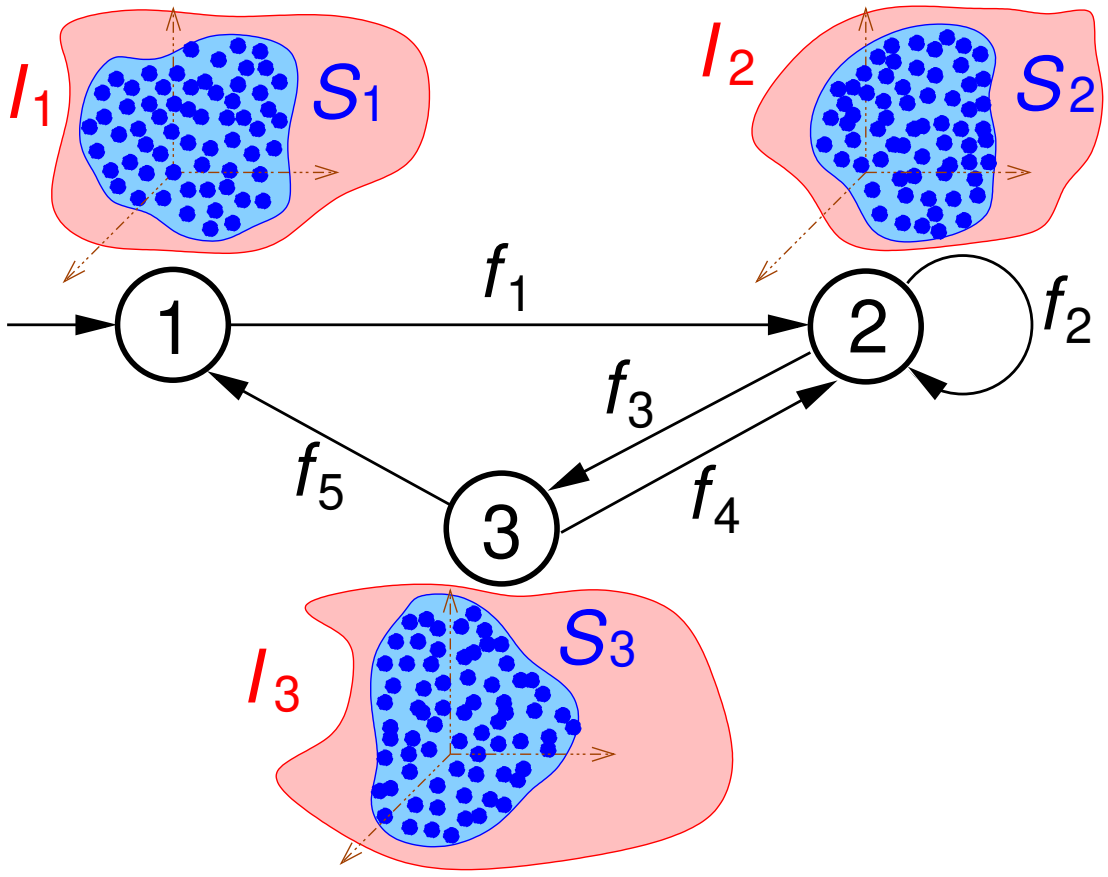

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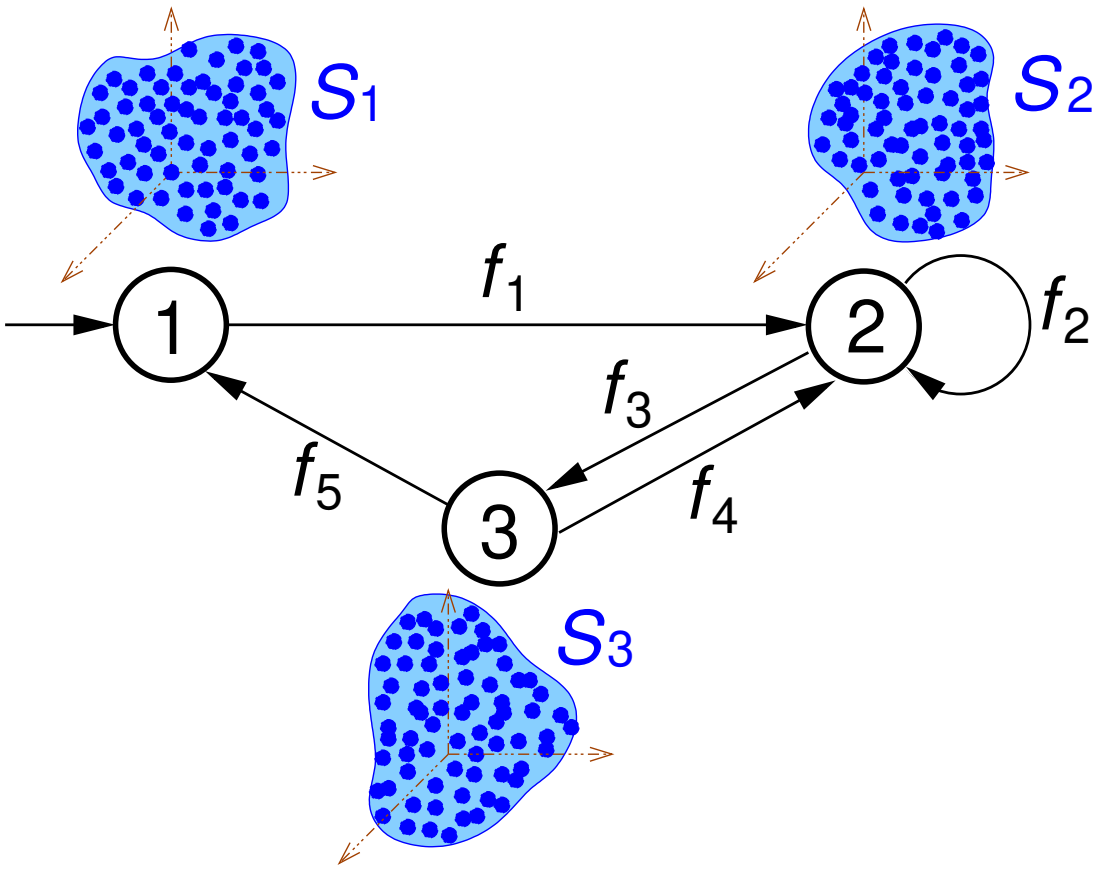

$\langle S_1, S_2, S_3 \rangle$ is **always** an inductive invariant

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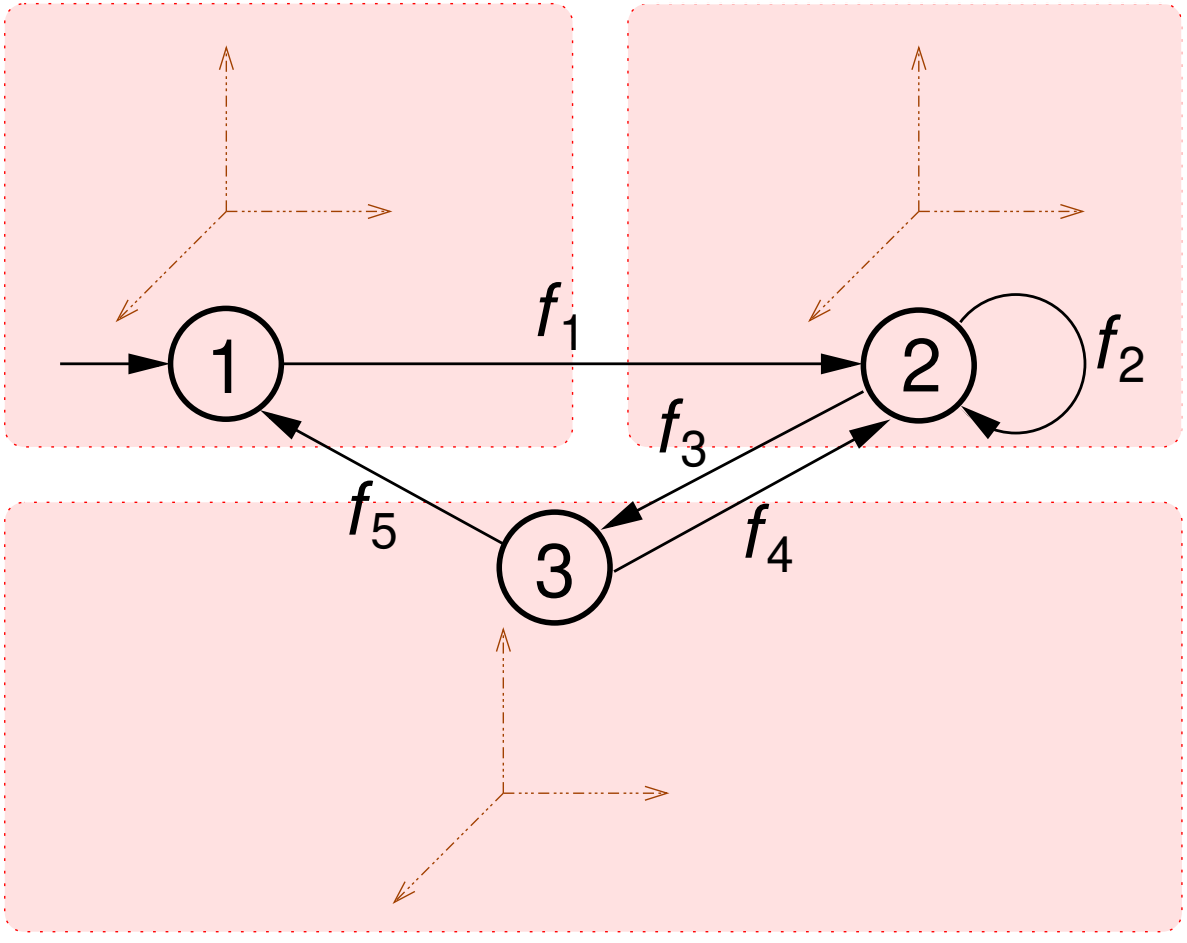

$\langle \mathbb{R}^3, \mathbb{R}^3, \mathbb{R}^3 \rangle$ is also **always** an inductive invariant

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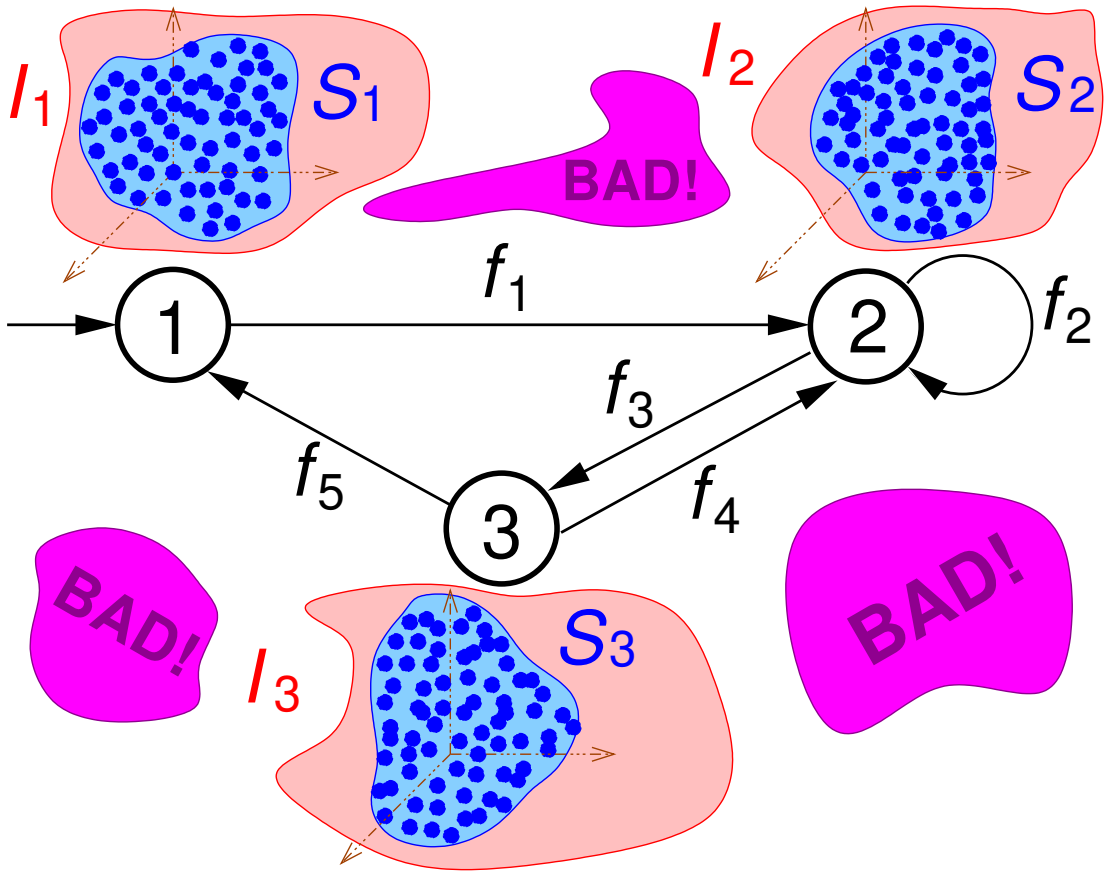

A good invariant is worth a thousand reachability queries!

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- Choose the right abstract domain
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  - Some domains always have ‘best’ (strongest, smallest) invariants, others not
- Compute an invariant!
  - Many eclectic methods: fixed-point computations, constraint solving, interpolation, abduction, machine learning, . . .
  - Some approaches require ‘widening’ to ensure termination
  - Other techniques invoke e.g. dimension or algebraic arguments
  - Often trade-off between precision and complexity . . .

A Menagerie of Abstract Domains

- **Intervals** [Cousot, Cousot 76], [Harrison 77]

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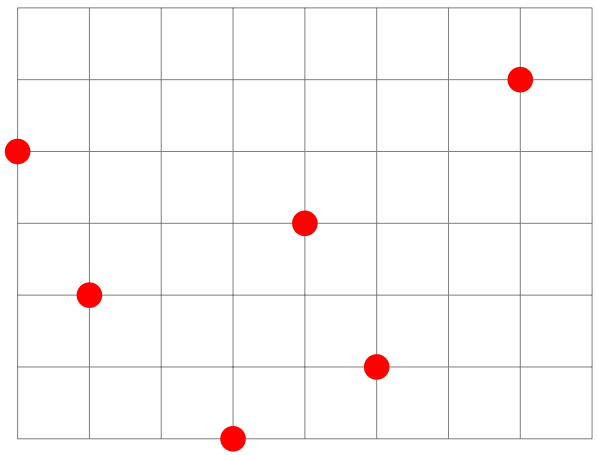
$$x^2 + y^2 + z^2 \leq 0 \quad \vee \quad x^2yz^5 - 3yz + 6 \geq 0$$

Comparing Abstractions

Original set:



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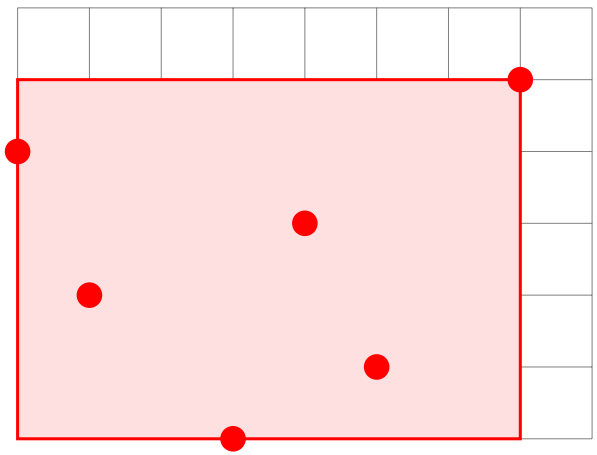
Interval abstraction:



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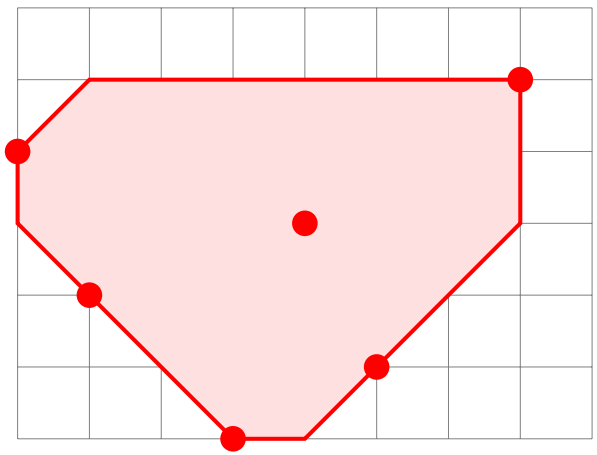
Octagonal abstraction:



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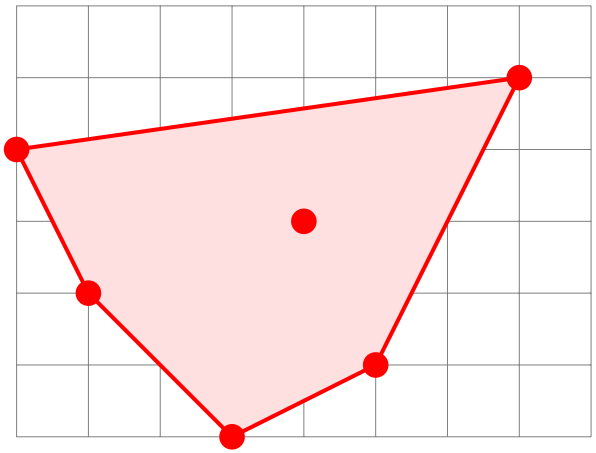
Polyhedral abstraction:



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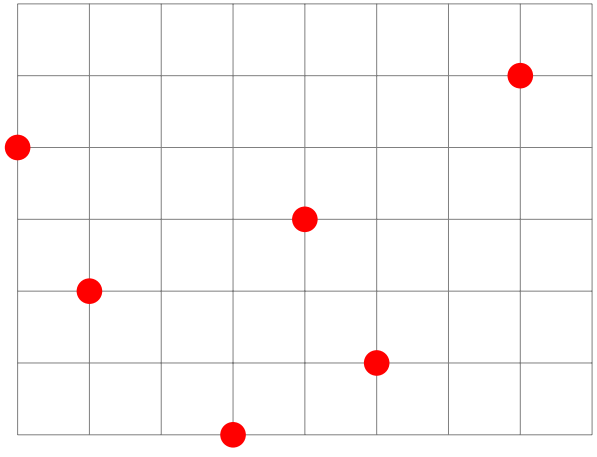
Algebraic/semialgebraic/semilinear abstraction :



Comparing Abstractions

$$\begin{array}{ccccccc} \text{Interval} & \leq & \text{Octagonal} & \leq & \text{Semilinear} & \leq & \text{Semialgebraic} \\ & & & & \vee\vert & & \vee\vert \\ & & & & \text{Linear} & \leq & \text{Algebraic} \end{array}$$

Why Linear Invariants Are Not Enough

$s := 0;$
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Or equivalently:

$$p(s, x) = 2s - x^2 - x = 0$$

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$x := 3;$

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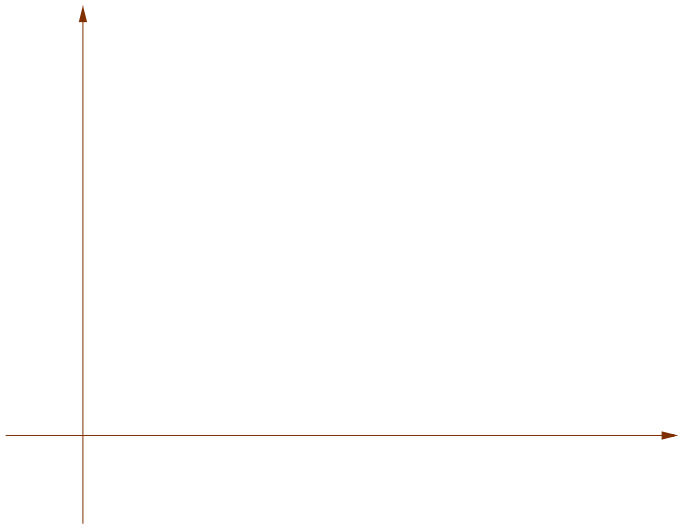

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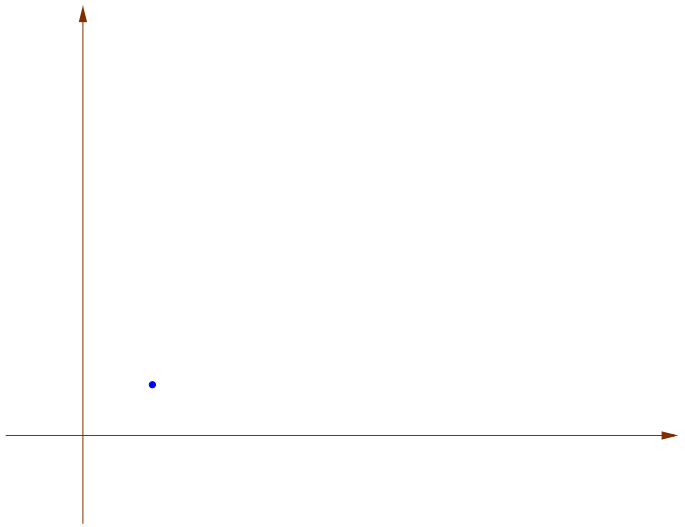

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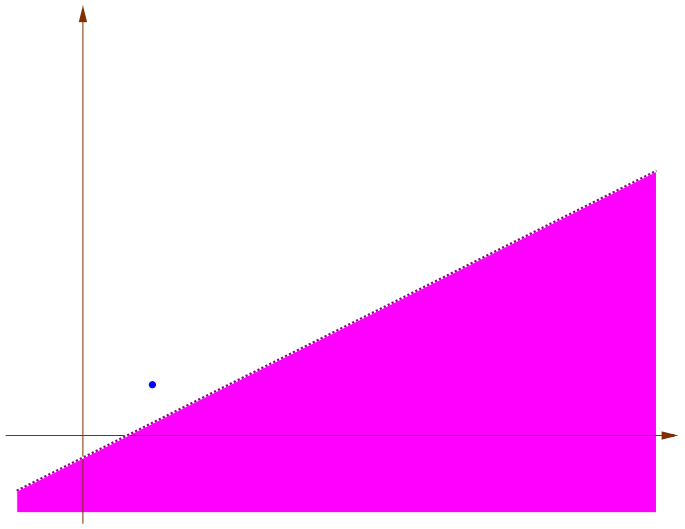

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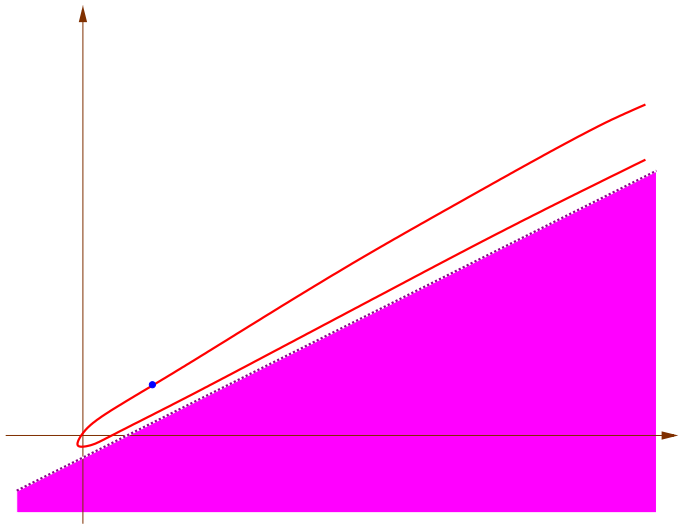

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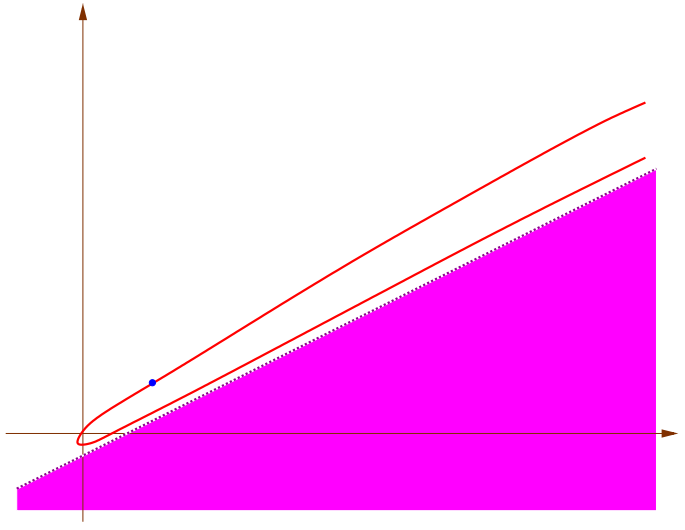

Polynomial invariant: $9x^2 - 24xy - x + 16y^2 + y = 0$

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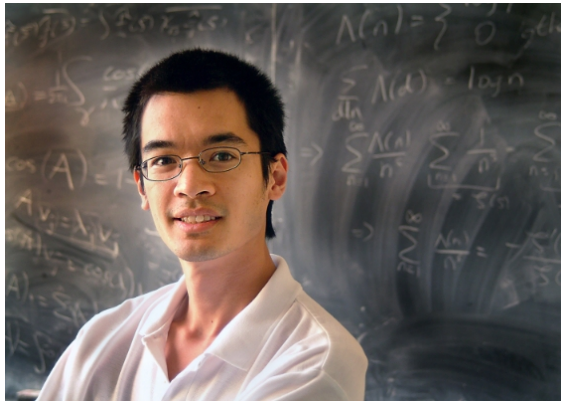
Deciding termination of simple linear loops is open!

"It is faintly outrageous that this problem is still open; it is saying that we do not know how to decide the Halting Problem even for 'linear' automata!"

Terence Tao



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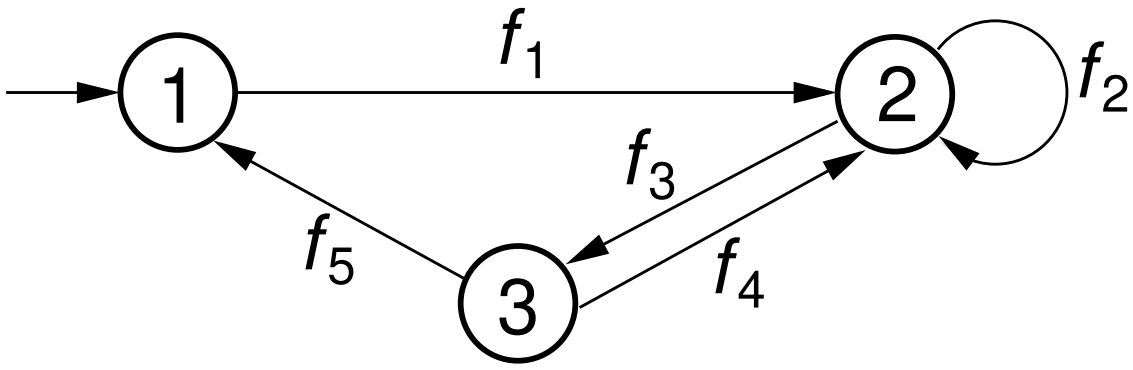
"We started this work hoping to vindicate forty years of research on heuristics by showing that the existence of polyhedral inductive separating invariants in a system with transitions in linear arithmetic (integer or rational) is undecidable."
David Monniaux

What Are Affine Programs?



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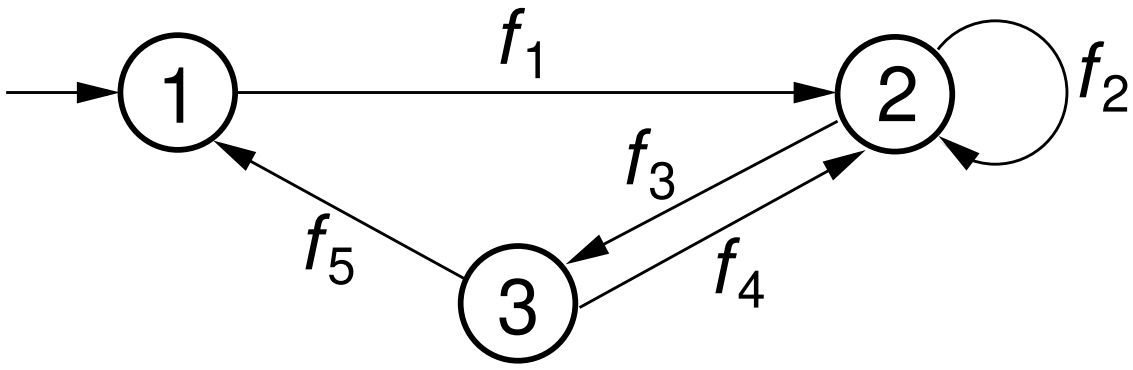

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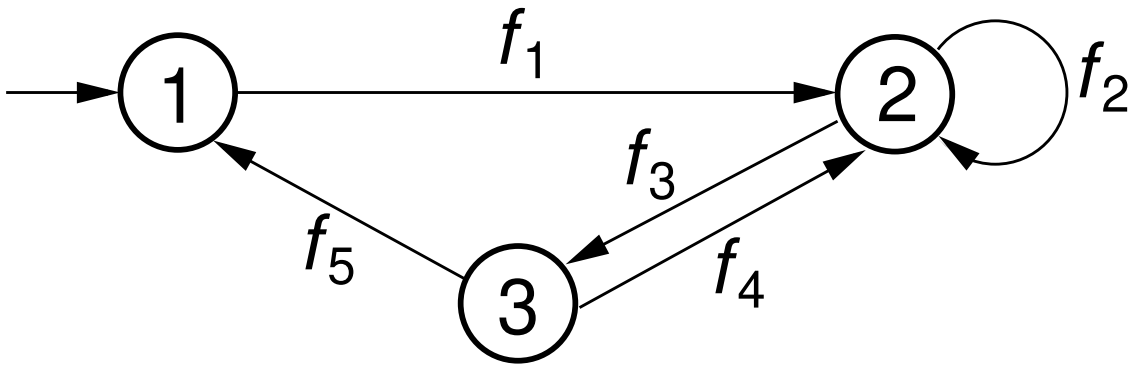

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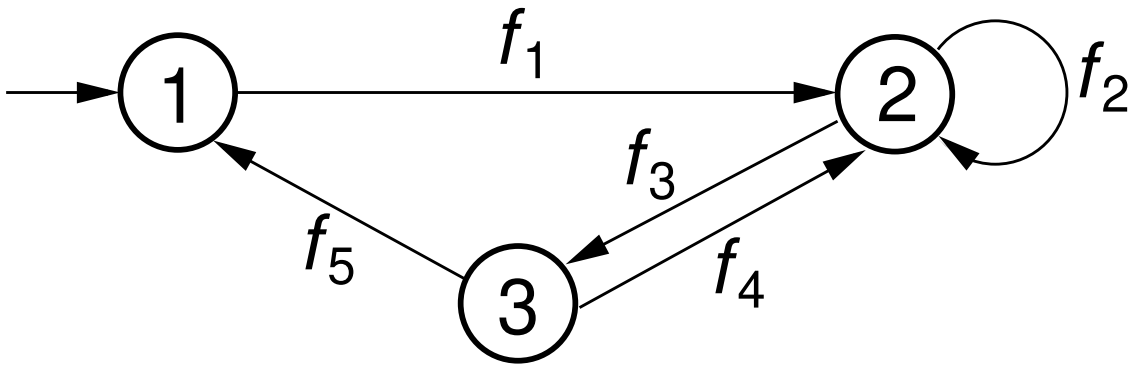

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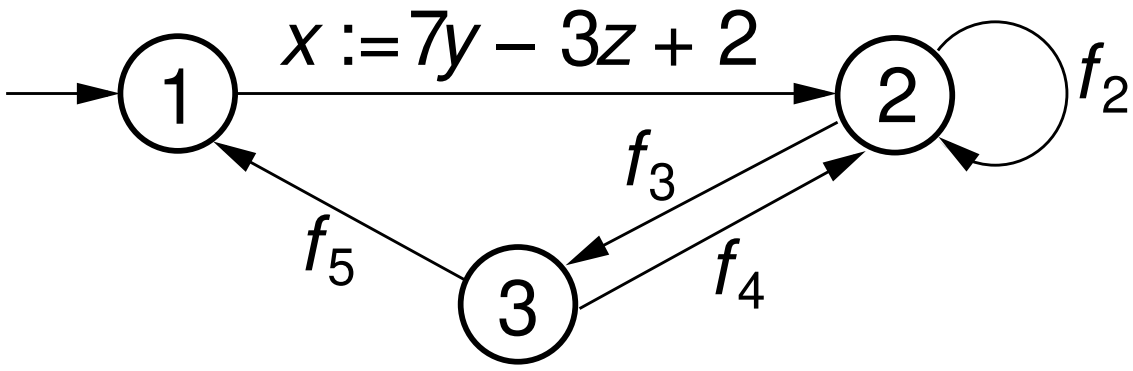

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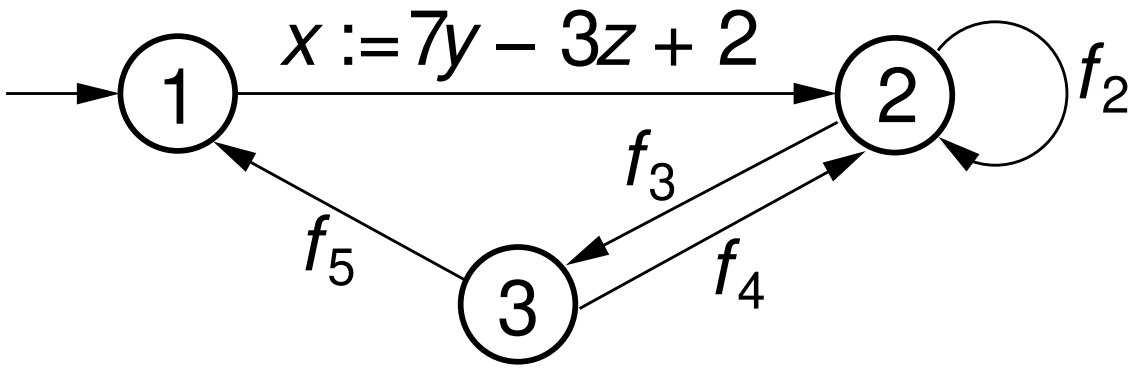

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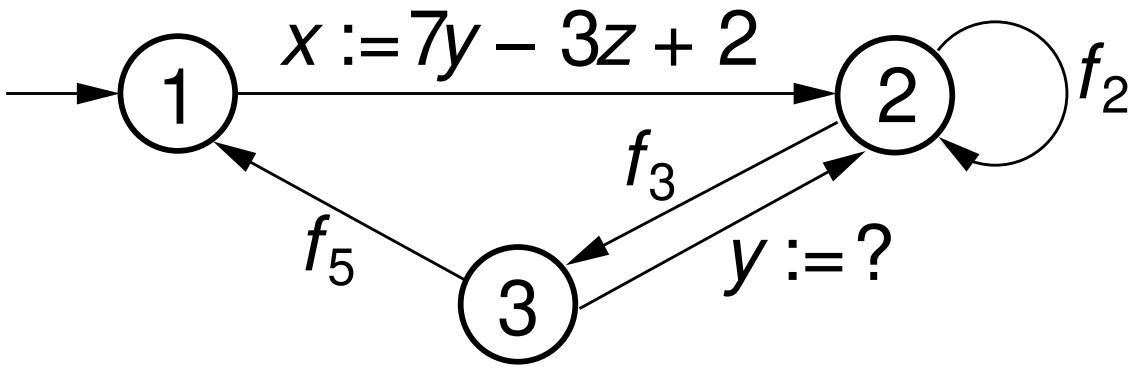

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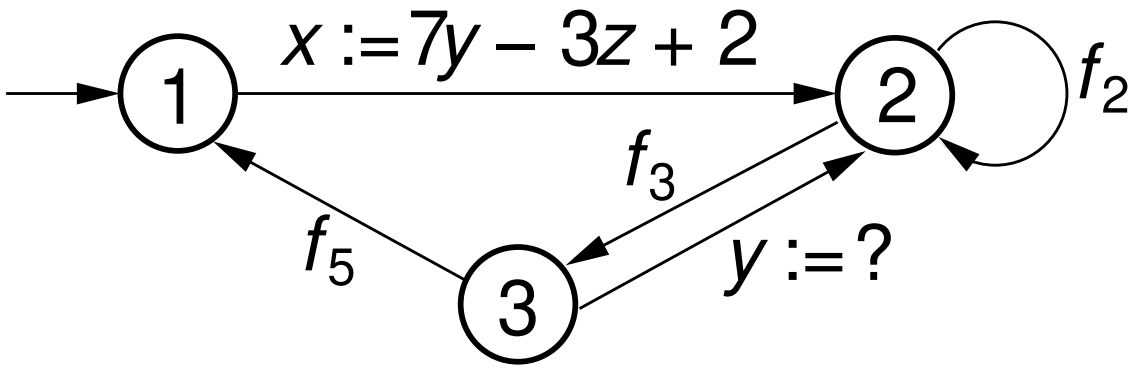

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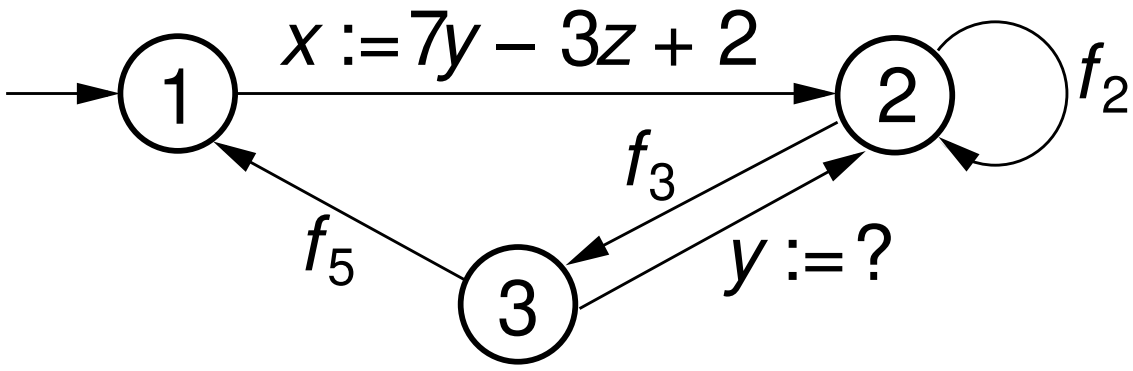

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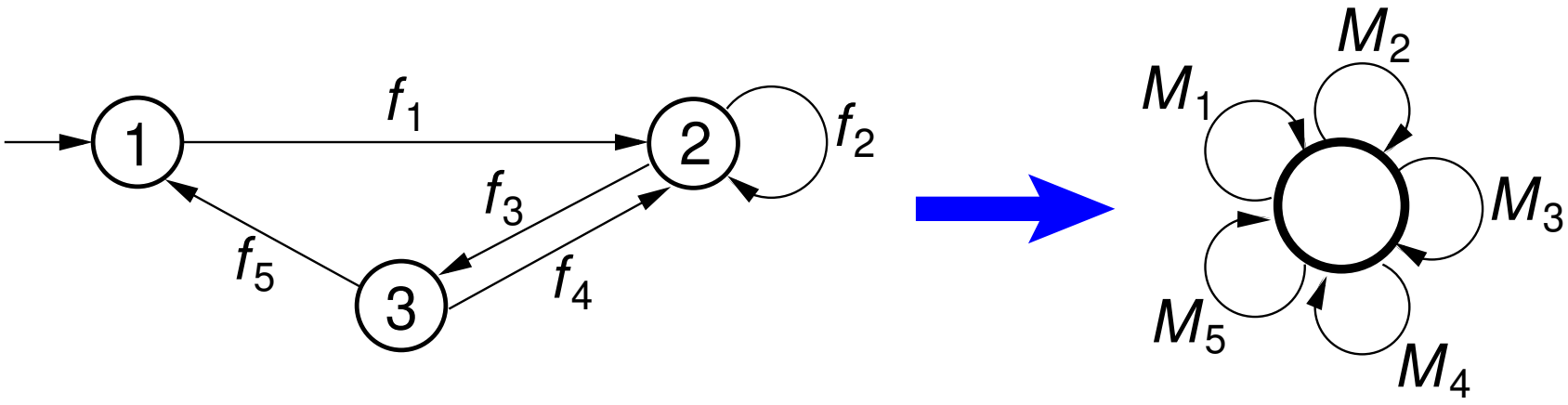
Affine programs:

- can **overapproximate** more complex programs
- already cover a range of existing formalisms, e.g. **probabilistic** / **quantum** / **quantitative automata**, ...

From Affine Programs to Linear Semigroups



each $M_i \in \mathbb{Q}^{d^2}$

Some Hard Problems for Linear Semigroups

Theorem (Markov 1947)

There is a fixed set of 6×6 *integer matrices* $M_1, \ldots, M_k$ *such that the membership problem "*$M \in \langle M_1, \ldots, M_k \rangle$*?" is* **undecidable**.

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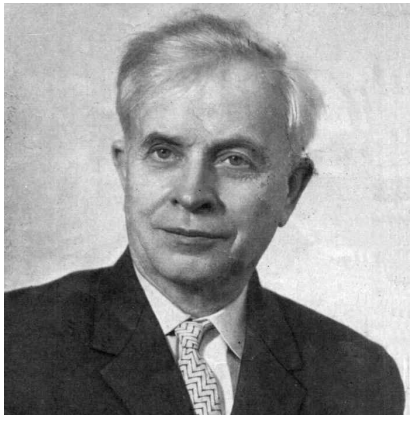

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Theorem (Paterson 1970)

The mortality problem is **undecidable** *for* 3×3 *matrices.*

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- **Semialgebraic sets** [Bagnara et al. 05]

$$x^2 + y^2 + z^2 \leq 0 \quad \vee \quad x^2yz^5 - 3yz + 6 \geq 0$$

State of the Menagerie

- ✓ **Intervals** [Cousot, Cousot 76], [Harrison 77]

$$x \in [0, 4] \quad \wedge \quad y \in [2, \infty)$$

- ✓ **Octagons** [Miné 06]

$$x + y - 2 \leq 2 \quad \wedge \quad x \leq 3 \quad \wedge \quad y - x \leq 1$$

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Karr's Algorithm, Acta Informatica 76

Affine Relationships Among Variables of a Program*

Michael Karr

Received May 8, 1974

Summary. Several optimizations of programs can be performed when in certain regions of a program equality relationships hold between a linear combination of the variables of the program and a constant. This paper presents a practical approach to detecting these relationships by considering the problem from the viewpoint of linear algebra. Key to the practicality of this approach is an algorithm for the calculation of the "sum" of linear subspaces.

Theorem (Karr 76)

There is an algorithm which computes, for any given affine program over $\mathbb{Q}$, its strongest linear inductive invariant.

Smallest Algebraic Invariants

Theorem (Hrushovski, Ouaknine, Pouly, Worrell 18)

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Theorem (Hrushovski, Ouaknine, Pouly, Worrell 18)

The presence of guards or polynomial transitions makes the problem undecidable.

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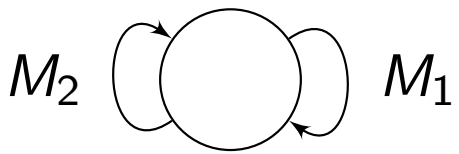

- Only two transitions required (matrices of size 336).
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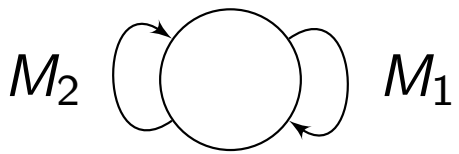

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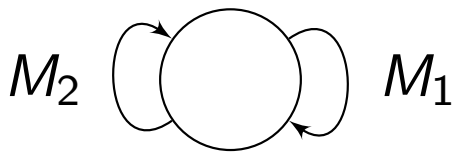

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Invariant of the form: $I = S \cup F$
S is a simple safe set.
F is a finite number of points.

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While $x \notin Bad$ **do**
$x := Mx$

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Theorem

The problem of the existence of semilinear safety invariants for while loop is decidable.

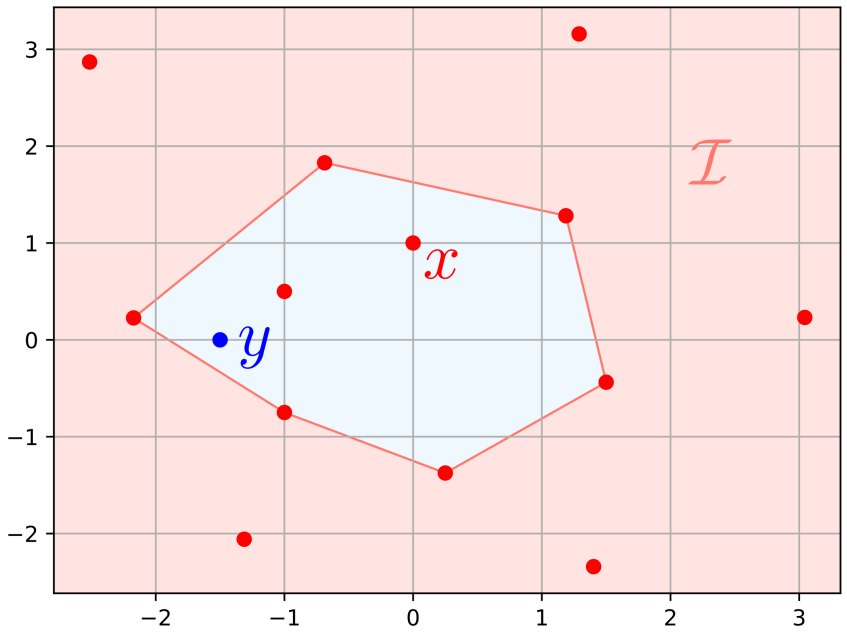
- Build the Jordan normal form;
- Build invariant if there exist "simple" eigenvalues;
- Prove the absence of non-trivial semilinear invariant otherwise.

Case $|\lambda| > 1$

Starting in x

Bad $= y$

Sequence in the eigenspace is a diverging spiral



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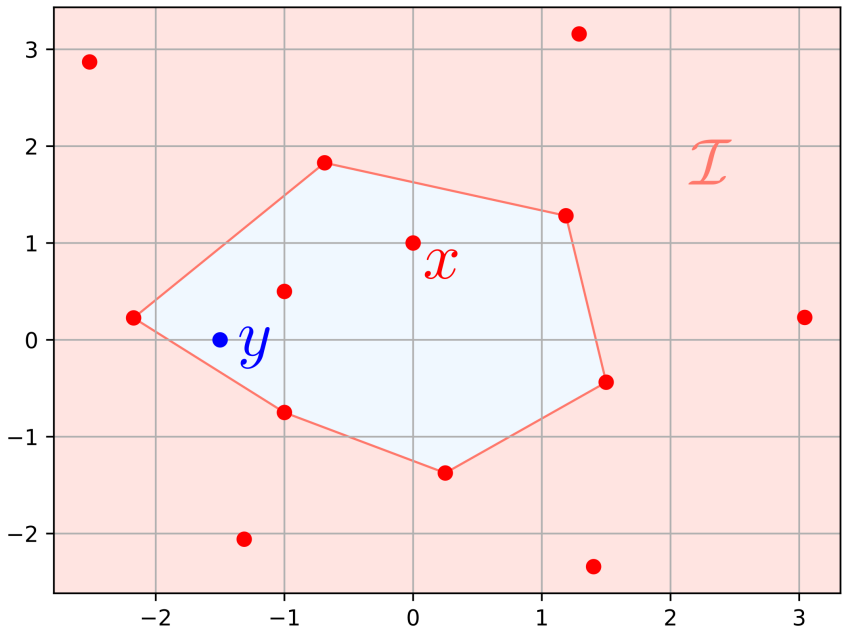

Most difficult case: modulus 1 and not root of unity.

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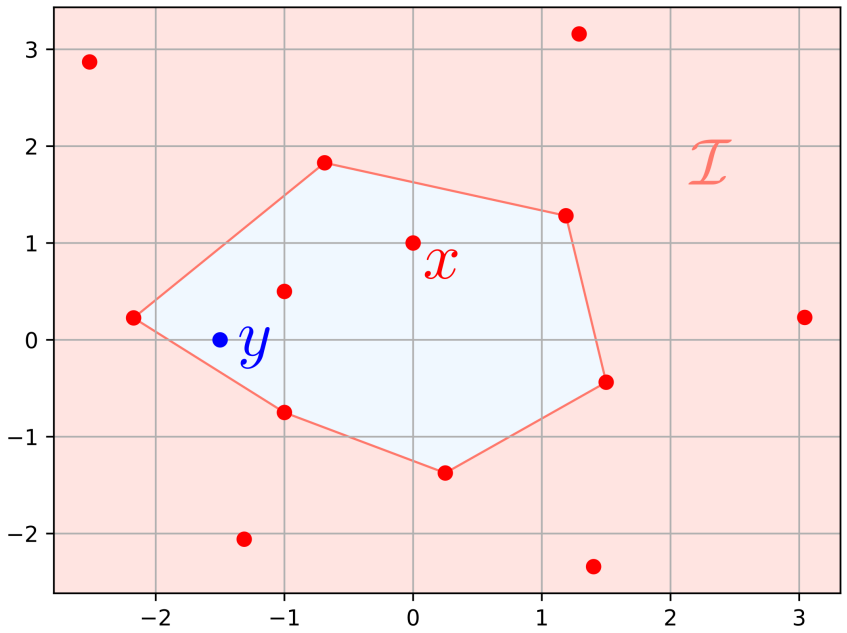

Most difficult case: modulus 1 and not root of unity.

$\rightarrow$ Only semilinear invariant given by relations between eigenvalues

What about Convex Invariant?

- ✓ simplicity of representation and implementation
- ✓ algorithmic tractability and scalability
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Theorem (Monniaux 19)

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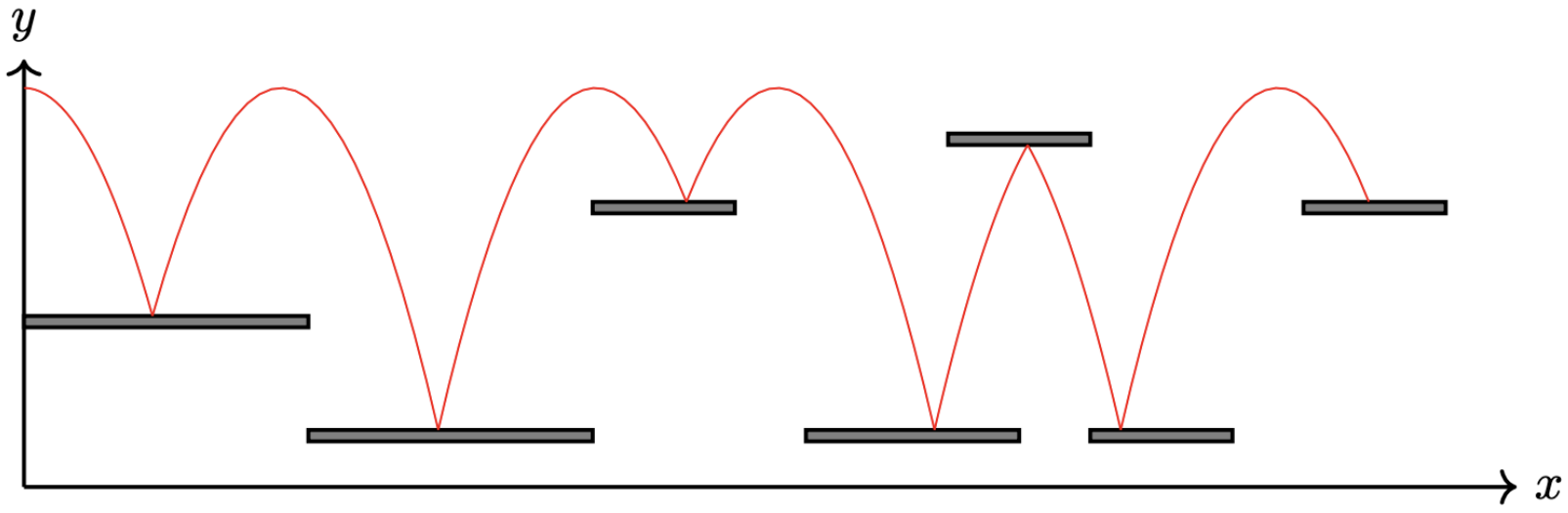
The general case remains open and challenging.

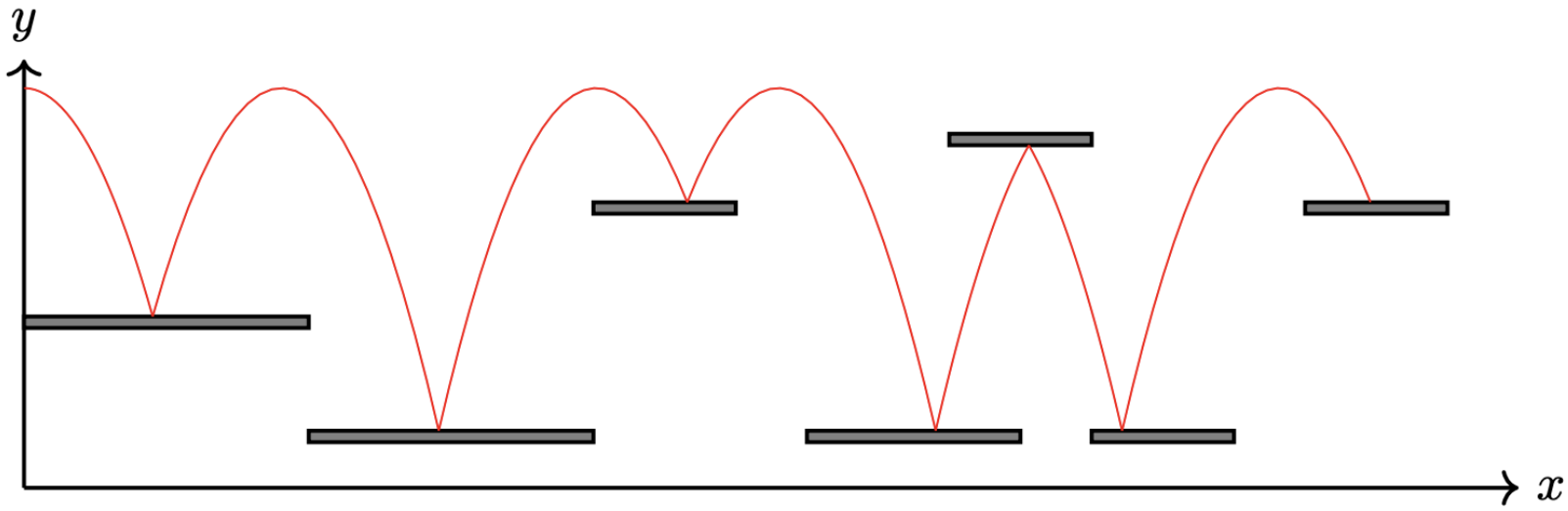
Ongoing Research Programme

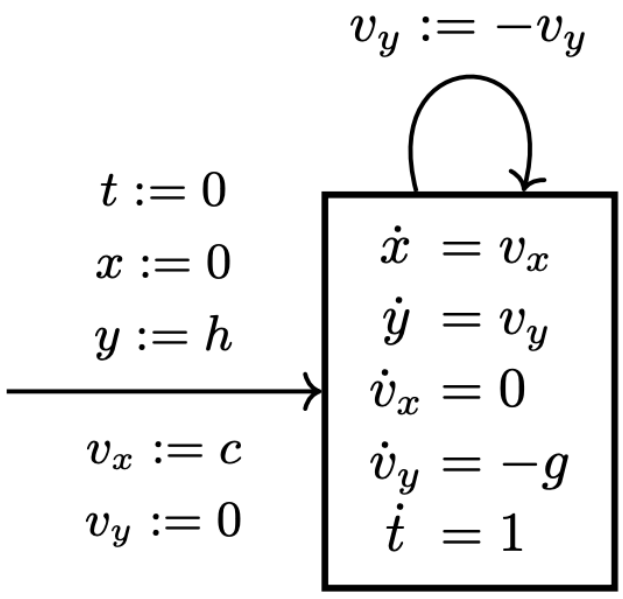
- The Monniaux Problem for convex invariants
- Orbit-finiteness for *polynomial* programs
- Algebraic and semialgebraic invariants for continuous dynamical systems & hybrid automata

A Bouncing Ball

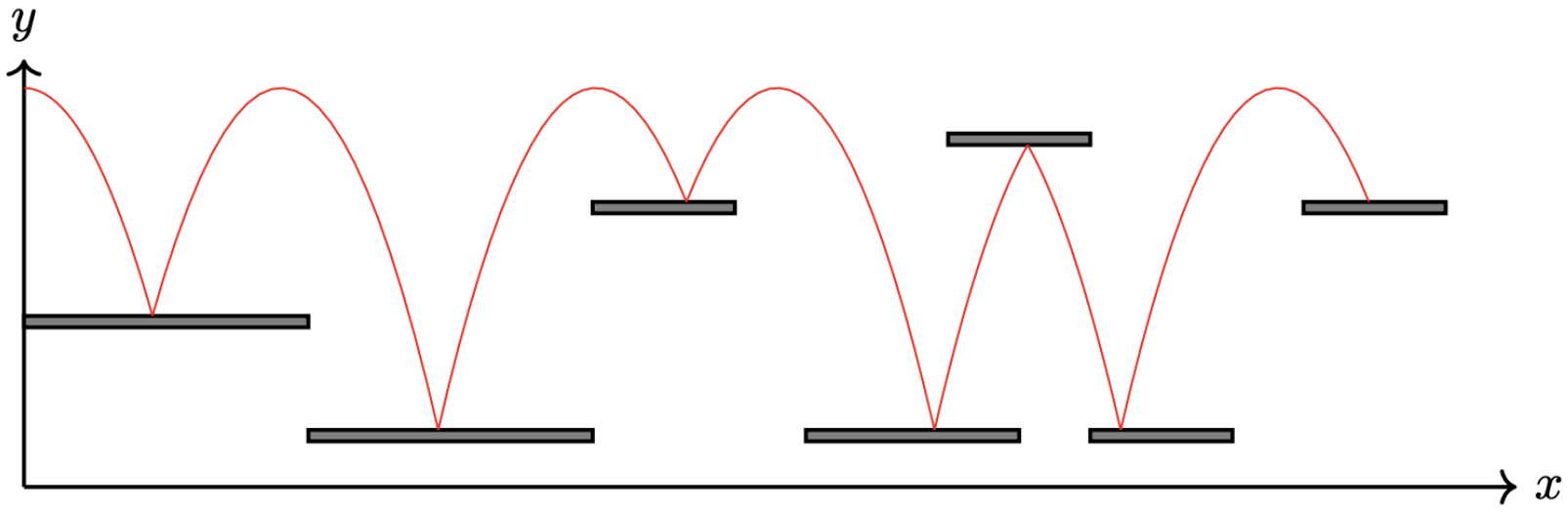


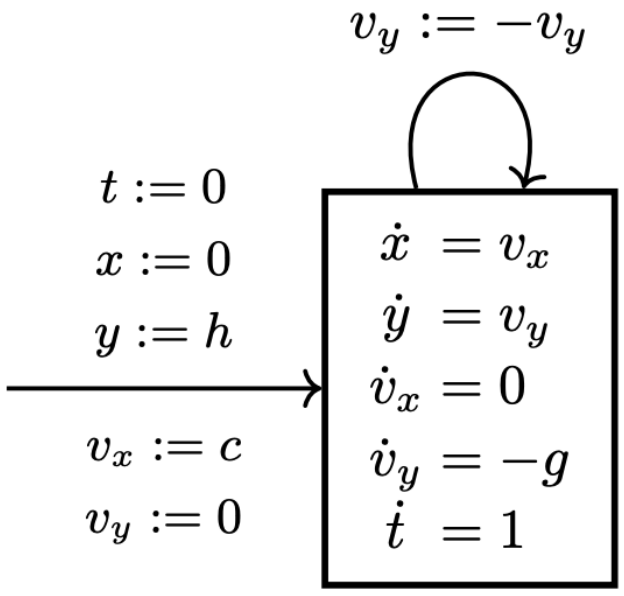
A Linear Hybrid Automaton (LHA)





Strongest Algebraic Invariants for LHA





$v_x = c$

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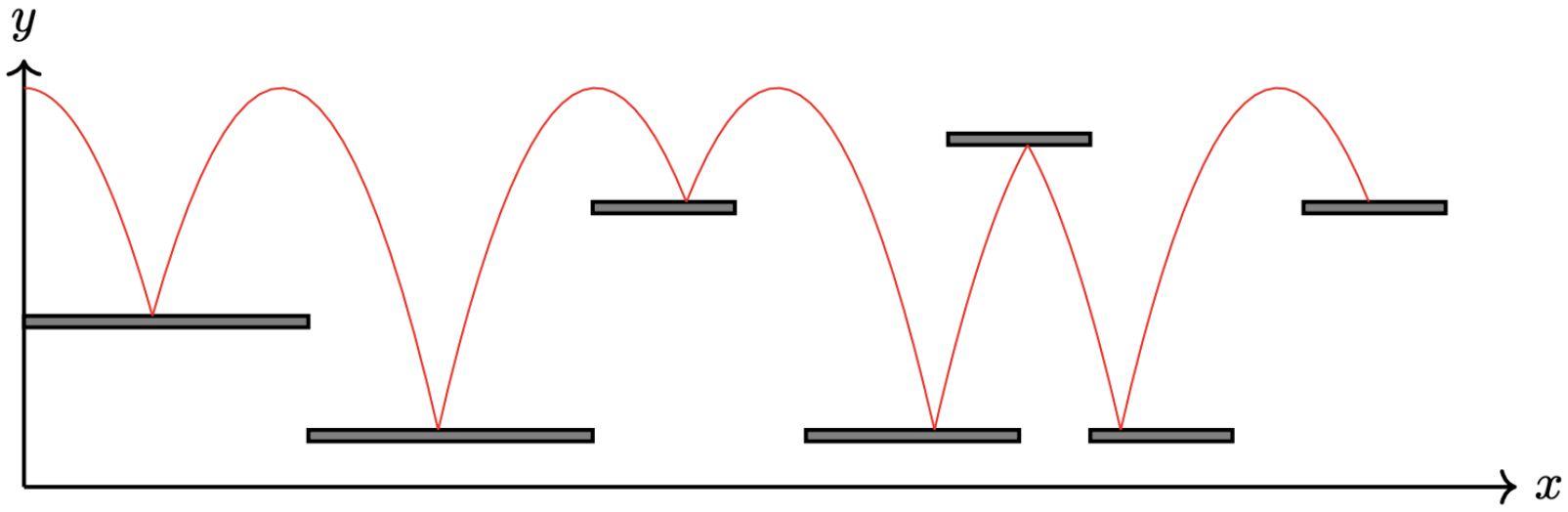

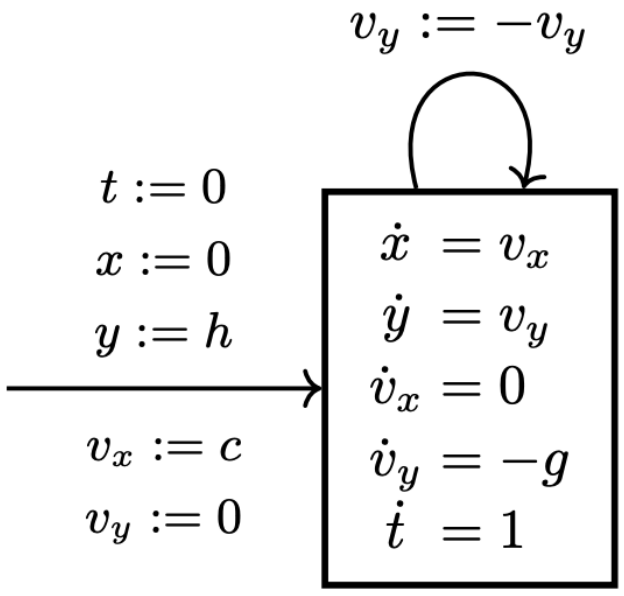

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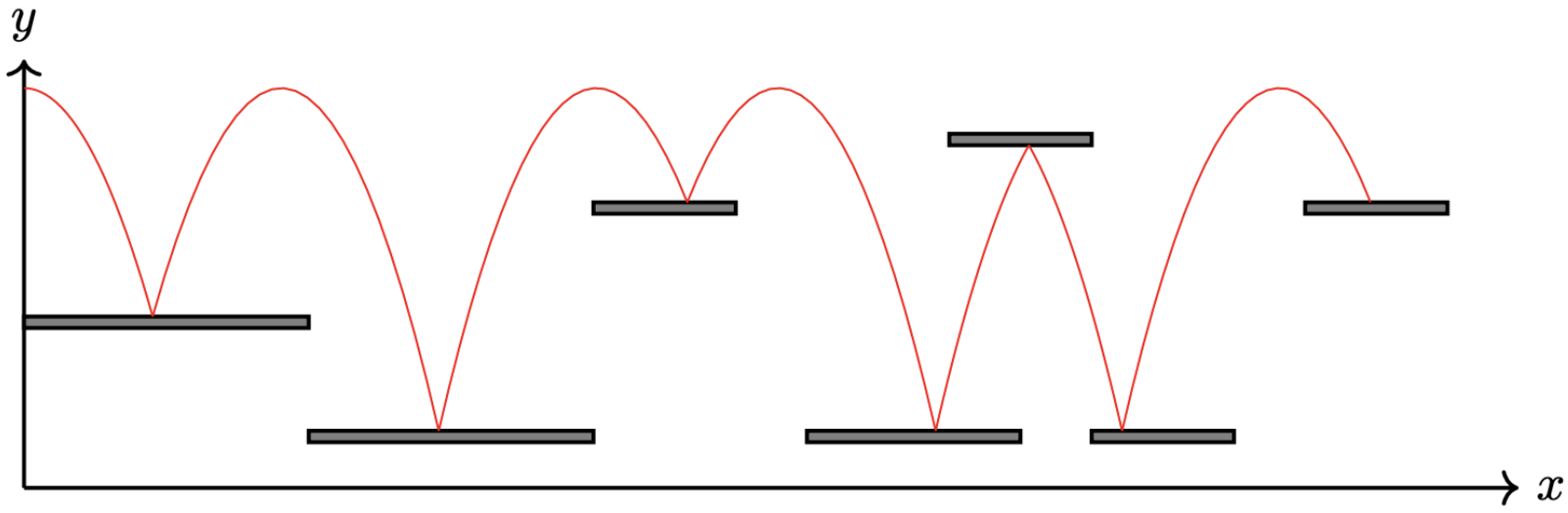

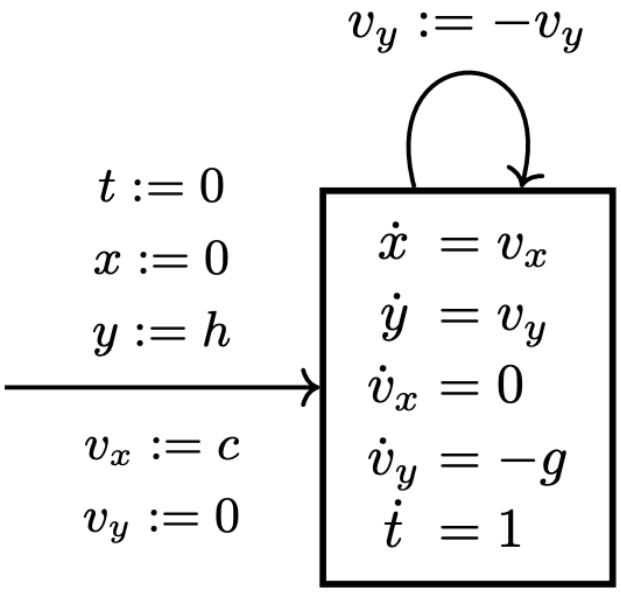

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$v_y{}^2 + 2g(y - h) = 0$